

# All smooth four-dimensional Schoenflies balls are geometrically simply connected

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## 1 Introduction

This paper will provide a complete proof for the result stated in the title above, namely the following theorem.

**Theorem 1.** *Any smooth 4-dimensional Schoenflies ball, which we will denote by  $\Delta_{\text{Schoenflies}}^4$  or simply  $\Delta^4$ , is geometrically simply-connected (G.S.C.).*

Our proof relies heavily on the celebrated work by Barry Mazur [Ma] from (1958), of which I remind here what is necessary for us right now.

What Barry showed, in the DIFF context is that, for any  $n$ , if  $\Delta^n \equiv \Delta_{\text{Schoenflies}}^n$ , then

$$\Delta^n - \{\text{a boundary point}\} \underset{\text{DIFF}}{=} B^n - \{\text{a boundary point}\}.$$

Soon after 1958, S. Smale, M. Kervaire and J. Milnor [Ke-M], [S] improved this, for all  $n \neq 4$  showing that in that case  $\Delta^n \underset{\text{DIFF}}{=} B^n$ .

Of course, for  $n \leq 3$  this last result was already known, and less high-power technology was necessary. For  $n = 3$ , this was known through the work of J.W. Alexander in the nineteen twenties, while for  $n = 2$  any good, strong enough proof of the fundamental conformal representation theorem is complex analysis (sometimes referred to as the “Riemann mapping theorem”), tells us what we need. But then, in four dimensions, the only known thing, as of today, is just Barry’s old result i.e. that

$$(0) \quad \Delta_{\text{Schoenflies}}^4 - \{\text{a boundary point}\} \underset{\text{DIFF}}{=} B^4 - \{\text{a boundary point}\},$$

where of course  $B^4$  means the standard 4-ball. One should notice that the (0) is actually equivalent to

$$(1) \quad \text{int } \Delta^4 \underset{\text{DIFF}}{=} R_{\text{standard}}^4$$

and this is what we will actually use in this paper. [The implication (0)  $\Rightarrow$  (1) is obvious, and for the implication (1)  $\Rightarrow$  (0) it is sufficient to show the following FACT. Up to diffeomorphism, there is a unique way to add a copy of  $R^3$  as boundary of  $R^4$ . Here is, in a nutshell, the proof of the FACT. Let  $Y^4 = \{R^4 \text{ with a copy of } R^3 \text{ glued at infinity, as a boundary}\}$ . If we take a tubular neighbourhood of  $R^3$  in  $Y^4$ , its generic fiber defines a PROPER embedding,  $[0, \infty) \xrightarrow{\varphi} R^4$ . It is not hard to show that the diffeomorphism

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type of the pair  $(R^4, \varphi[0, \infty))$  determines the diffeomorphism of the non compact manifold with non-empty boundary  $Y^4$ .

At this point, we can invoke the well-known fact that in dimensions  $n \geq 4$  there are no wild Artin-Fox arcs, and this clinches the proof that  $(1) \Rightarrow (0)$ .]

We shall denote now by  $\Delta^2$  the 2-skeleton of  $\Delta^4$  and here is now a statement which is equivalent to our Theorem 1.

**Theorem 2.** *The regular neighbourhood of  $\Delta^2 \subset \Delta^4$ , call it  $N^4(\Delta^2)$ , is G.S.C.*

Our arguments, in this paper, will actually provide a proof for Theorem 2.

The rest of this introductory section deals with a bit of history. In July 2006 I have put on line the paper

“On the 3-dimensional Poincaré Conjecture and the 4-dimensional Schoenflies Problem”.

[ArXiv.org/abs/math.GT/0612554]

There is a big organic connection between that ArXiv paper, which was itself a fast survey of a never published, actually never typed, much longer manuscript, called “Po V-B”, and our present paper. The Po V-B and its ArXiv summary, dealt with a simultaneous proof of the following two items, namely

- i) The proof that, for any homotopy 3-ball  $\Delta^3$ , we have  $\Delta^3 \times I \in \text{GSC}$ . The proof was starting from a previous result of mine [Po1], [Po2], which is the following: “The open DIFF 4-manifold  $\{\Delta^3 \times I \# \infty \# (S^2 \times D^2)\}$ , with all the boundary removed, is GSC.” Actually, the proof required not only the [Po1], [Po2] mentioned above, but also some other previous papers by the same author, which I will no longer mention here. But the interested reader may find a synthetical account of these things in [O-Po-Ta]. Then, with very much the same techniques, the Po V-B did contain the following item.
- ii) The proof that  $\Delta_{\text{Schoenflies}}^4$  is GSC. This time, exactly as the [Po1], [Po2] above was used in the proof of i), one uses Barry’s result (0) (or (1)).

If from the ArXiv 2006 paper, as it stands, one carefully removes all the references to 3<sup>d</sup> Poincaré Conjecture, then what one gets is a pretty accurate description of the *plan* of this present work.

The point is that, although the starting points of i) and ii) above look quite different, the arguments for proving them are, in more than one way, very similar. When one looks into the seams, then some points are easier for i) and harder for ii), while for others quite the converse is true. For instance, while  $\Delta^3$  is of codimension one inside  $\text{int}(\Delta^4 \times I \# \infty \# (S^2 \times D^2))$ , the  $\Delta_{\text{Schoenflies}}^4$  is codimension zero in  $\Delta^4 \cup \partial\Delta^4 \times [0, 1) \stackrel{\text{DIFF}}{=} R^4$ , which makes things a bit harder for  $\Delta^4$  than for  $\Delta^3 \times I$ .

But then, while  $R^4 - \Delta^4$  is essentially product, the

$$\text{int}(\Delta^3 \times I \# \infty \# (S^2 \times S^2))$$

contains those infinitely many  $S^2 \times D^2$ ’s, source of some technical complications. But then, also, when viewed from a higher vantage point these are rather minor issues and, more seriously, a certain specific argument for ii) was painfully lacking for a long time. Then, in the Spring 2003, while I was visiting Princeton University, during some homeric working sessions with Dave Gabai and Frank Quinn, I had sort of an illumination and I saw the right ingredient which I was missing before. This ingredient is described in the 2006 ArXiv paper and it occurs here as formula (21.A) in the text which follows.

Anyway, nine years later, when I looked again into these things, in the end of 2014 and the beginning of 2015, then I realized that, for some reasons which may be were understandable in those early years, when I wrote the PO V-B, I had sort of blinders, at that time, failing to see the big forest which was looming behind those nearest trees. That made that my own re-reading, of that old paper of mine, the old Po V-B, was a very painful and rather heavy affair.

The fact is that, years ago, while I was writing that Po V-B I did not know where I would get, until towards the end, and the written paper keeps the traces of the various unnecessary detours and blind alleys. Now I knew exactly what the correct scheme was, and so life was much easier. But once I understood well the whole idea, I thought things over again, and rather than following blindly the old Po V-B, I started by disentangled the Schoenflies part from my Poincaré papers, which will no longer be mentioned here from now on, and thereby I managed to vastly simplify and also shorten the arguments for Schoenflies, which I extracted from the Po V-B. I found a lot of simplifying tricks too.

Of course, there is now also the obvious issue of getting from the Theorem 1 stated above, to the full DIFF  $4^d$  Schoenflies, and the additional step which is still necessary, in order to achieve that, is to show that any smooth  $4^d$  Schoenflies ball  $\Delta_{\text{Schoenflies}}^4$ , which is in GSC is also standard, i.e. diffeomorphic to  $B^4$ . That is the object of current joint work with Dave Gabai and I will not discuss it here, except for the following little comment. The fact that  $\partial\Delta_{\text{Schoenflies}}^4 = S^3$  is NOT used in the present paper. This is quite natural, once it is understood that the same technology serves for proving both that  $\Delta_{\text{Schoenflies}}^4 \in \text{GSC}$  AND that  $\Delta^3 \times I \in \text{GSC}$ . But then, the fact that  $\partial\Delta^4 = S^3$  should be used big in the joint paper with Dave.

I want to end this introduction with some philosophical-speculative thoughts. The Schoenflies  $4^d$  DIFF problem is strongly connected to the following two outstanding questions in  $4^d$  topology: the DIFF  $4^d$  Poincaré Conjecture and the big open gap or chasm, which exists in  $4^d$ , and only in  $4^d$ , between the categories DIFF and TOP.

There are many ways in which our Schoenflies problem is connected with the  $4^d$  Poincaré Conjecture, and I will only mention them here.

If one assumes the full  $4^d$  DIFF Schoenflies, then it easily follows that every smooth DIFF homotopy sphere  $\sum^4$  which is such that  $\sum^4 - \{\text{pt}\} \stackrel{\text{DIFF}}{=} R_{\text{standard}}^4$ , also comes with  $\sum^4 \stackrel{\text{DIFF}}{=} S^4$ . For more on this kind of topics, see also the short paper [Po3].

I will not say more, here and now, concerning that big gap, unbridgeable by algebraic topology, between DIFF and TOP, in dimension four, and only in dimension four.

By now years ago, already, I had numerous discussions with Dave Gabai concerning the issues treated in the present paper, which would have probably never existed without his friendly help. I wish to thank him warmly here. Many thanks are also due to Frank Quinn for the many very helpful discussions we had, and to Louis Funar to whom I have lectured about the matters presented here and who came with very useful questions and comments.

The present paper owes a lot to the long continuous connection of the author with the IHES. And then, last but not least, I wish to thank Cécile Gourgues for the typing and Marie-Claude Vergne for the drawings.

## 2 Geometric preliminaries and the doubling process

From now on,  $\Delta^4$  will mean the Schoenflies smooth 4-ball  $\Delta_{\text{Schoenflies}}^4$  and we will denote by  $X^4 \stackrel{\text{DIFF}}{=} R^4$  the interior of  $\Delta^4$ , or some chosen cell-division of it. We will work with the following collection of telescopically nested spaces

$$(2) \quad \Delta^4 = \Delta_{\text{small}}^4 \subset \Delta^4 \cup \partial\Delta^4 \times [0, 1] \subset \Delta_1^4 \equiv \Delta^4 \cup \partial\Delta^4 \times [0, 1],$$

where the middle  $\Delta^4 \cup \partial\Delta^4 \times [0, 1]$  is, up to diffeomorphism, our  $X^4$  while  $\Delta_1^4$  is a larger copy of  $\Delta^4$ .

Here, of course,  $\partial\Delta^4 = S^3$  and from now on,  $X^4$  will mean the  $\Delta^4 \cup \partial\Delta^4 \times [0, 1] = R^4$  in (2), or some specific cell-decomposition of it.

**Lemma 3.** *Let  $Y^4$  be a smooth compact bounded 4-manifold, for which by analogy to (2), we consider the telescopic system*

$$(2.1) \quad Y^4 = Y_{\text{small}}^4 \subset Y_{\text{small}}^4 \cup \partial Y^4 \times [0, 1] \equiv Y_1^4.$$

We also assume that there exists a system of smoothly embedded discs

$$(3) \quad \left( \sum_1^P D_i^2, \sum_1^P \partial D_i^2 \right) \subset (\partial Y^4 \times [0, 1], \partial Y^4 \times \{0\} = \partial Y_{\text{small}}^4),$$

which are in **cancelling position** with the 1-handles of  $Y_{\text{small}}^4$ . Then  $Y^4$  is geometrically simply-connected, we will write GSC.

By the “cancelling position” above, we mean that we can index the 1-handles of  $Y^4$  as  $H_1^1, H_2^1, \dots, H_p^1$ , so that when we consider the geometric intersection matrix  $\partial D_j^2$  (exterior 2-handle)  $\cdot H_i^1$  (interior 1-handle), we have

$$\partial D_j^2 \cdot H_i^1 = \delta_{ji}.$$

The GSC concept makes sense both for cell-complexes and for smooth manifolds, and our hypothesis in this lemma can be also stated as  $Y_{\text{small}}^4 + \sum_1^P D_i^2$  is GSC OR

$$N^4 \left( Y_{\text{small}}^4 + \sum_1^P D_i^2 \right) \text{ is GSC.}$$

The argument which follows is elementary differential topology.

**Proof of Lemma 3.** For the collar  $\partial Y^4 \times [0, 1]$  we have two basic projections

$$\begin{array}{ccc} \partial Y^4 \times [0, 1] & \xrightarrow{\pi_0} & [0, 1]. \\ \downarrow \pi & & \\ \partial Y^4 = \partial Y_{\text{small}}^4 & & \end{array}$$

We may assume that the smooth map from  $2^d$  into  $3^d$   $\pi \Big| \sum_1^P D_i^2$  is generic, i.e. that this map is a generic immersion outside of a finite, set of isolated singularities, generically denoted by  $s \in \text{int } D^2$ , which are **Whitney’s umbrellas**. Notice that, because of the embedding condition in (3), the occurrence of clasps as double lines is excluded. For such a clasp there would be a contradiction between the  $\pi_0$ -values at the two boundary points. We will perform now the following steps which, provisionally, will add 1-handles to  $Y_{\text{small}}^4 \cup \sum_1^P D_i^2$ , to be killed later.

1) We eliminate the triple points, as follows. At each triple point (of  $\pi \Big| \sum_1^P D_i^2$ ) we have three branches which, from the viewpoint of the  $\pi_0$ -values at the triple point, are  $B^2$ (lowest),  $B^2$ (median),  $B^2$ (highest). We create a hole  $t$  inside  $B^2$ (lowest)  $\subset \sum_1^P D_i^2$  and, at the same time we add a 1-handle to  $N^4 \left( Y_{\text{small}}^4 \cup \sum_1^P D_i^2 \right)$ , by pushing the center of the hole  $t$  down to  $\partial Y_{\text{small}}^4$ , so that  $\partial(\text{hole } t) \subset \partial Y_{\text{small}}^4$ .

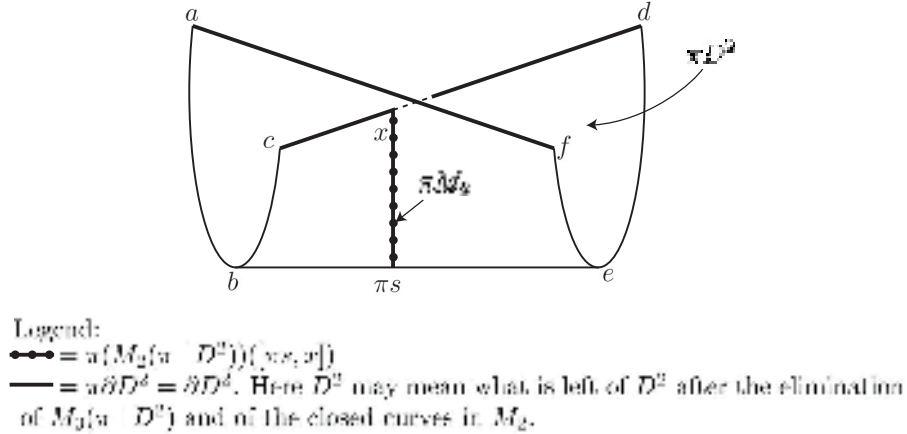
2) We can also eliminate the closed double curve, proceeding as follows: Start by noticing that each such curve  $C$  considered as a subset of  $M^2(\pi)$  consists either of two components  $C(\text{up})$ ,  $C(\text{down})$ , coming with  $s(C(\text{up})) = C(\text{down})$ , where  $s$  is the involution  $\sum_1^P D_i^2 \times \sum_1^P D_i^2 \supset M^2(\pi) \xrightarrow{s} M^2(\pi)$  gotten by exchanging the two factors, or of a single component endowed with the fixed point free involution  $C \xrightarrow{s} C$ , such that  $\pi(x) = \pi s(x)$ . In this last case we can find two disjoint arcs  $A(\text{up})$ ,  $A(\text{down})$ , exhausting  $C$ , with

$s A(\text{up}) = A(\text{down})$ . We can then push a  $p \in C(\text{down})$ , respectively a  $p \in A(\text{down})$ , all the way down to  $\partial Y_{\text{small}}^4$ , like we just did it for the  $B^2(\text{lowest})$  above. With this step, the double curves of  $M_2 \left( \pi \mid \sum_1^P D_i^2 \right)$  are out of the picture, at the price of our  $D_i^2$ 's acquiring the holes  $t$ .

So, at the price of changing our source into a bunch of discs with holes  $t$ , we have gotten rid both of triple points of  $\pi$  and of the closed curves of double points, which leaves us, as far as the immersion double points are concerned, only with RIBBONS and with the contribution of the double lines shooting out of the Whitney umbrellas, the two disjointed from each other (since now  $M_3 = \emptyset$ ).

The double lines connecting two Whitney umbrellas in head-on collision can be easily eliminated, without changing the topology of  $N^4 \left( Y_{\text{small}}^4 \cup \sum_1^P D_i^2 \right)$ . So, we may assume from now on that from each Whitney umbrella point, one of the two outgoing double lines hits the boundary. From each Whitney singularity  $s$  starts now a double line of  $\pi \mid \sum_1^P D_i^2$  hitting the boundary, like in the Figure 1.

This figure is at the target of the map  $\pi$ , in  $\partial Y_{\text{small}}^4$ .



**Figure 1.**

This figure lives in  $\partial Y_{\text{small}}^4$ . So the line  $[\pi s, x]$  joins the Whitney umbrella with  $\partial D^2 \subset \partial Y_{\text{small}}^4$ .

We will get rid of these last surviving  $s$ 's by the following steps:

- i) At the target, inside  $\partial Y_{\text{small}}^4$  we crush the arc  $[\pi s, x]$  (see Figure 1) to its endpoint  $x \in \partial D^2$ .
- ii) At the same time as this, at the level of the source  $D^2$ , we crush to a point the inverse image of  $[\pi s, x]$ , meaning exactly the set

$$\pi^{-1}([\pi s, x]) = [x, s] \cup [s, x'], \quad \text{with } x \in \partial D^2, \quad x' \in \text{int } D^2.$$

The topology of  $N^4 \left( Y_{\text{small}}^4 \cup \sum_1^P D_i^2 \right)$  stays unchanged, but obviously not the map  $\pi$ , which gets simplified.

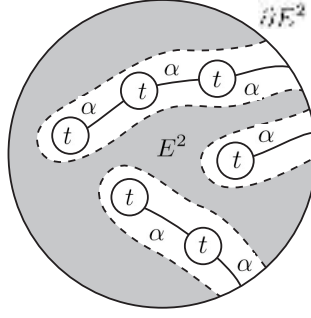
Notice that the conjunction of these steps i) and ii) changes the topology of  $\pi D^2$  but not the topology of  $D^2$  and, moreover, they leave intact the closed loop  $[a, b, c, d, e, f, a]$  in the Figure 1.

With all these operations, in which we have gotten rid of the triple points, of the closed curves of double points and of the Whitney umbrellas  $s$ , we have created a new situation, namely

(3.1) The  $\sum_1^P D_i^2$  occurring in (3) has been changed into an  $\sum_1^P E_i^2$  where each  $E_i^2$  is a disc with little holes  $t$ . Instead of (3), we have now

$$\left( \sum_1^P E_i^2, \sum_1^P \partial E_i^2 \right) \subset (\partial Y_{\text{small}}^4 \times [0, 1], \partial Y_{\text{small}}^4 \times \{0\}).$$

The  $\pi \mid \sum_1^P E_i^2$  is a **generic immersion**, the only double points of which are now RIBBONS. Moreover  $N^4 \left( Y^4 \cup \sum_1^P E_i^2 \right) = N^4 \left( Y_{\text{small}}^4 \cup \sum_1^P D_i^2 \right) + \{\text{a collection of vertical } 4^{\text{d}} \text{ 1-}\mathbf{handles}, \text{ which are in canonical bijection with the holes } t, \text{ each 1-handle being a } N^4(\text{vertical arc } [t, \pi t]) \subset \partial Y_{\text{small}}^4 \times [0, 1]\}.$   
End of (3.1)



**Figure 2.**

The disc with holes  $E^2$ , at the source of  $\pi$ , with holes  $t$  and with arcs  $\alpha$  joining them acyclically to  $\partial E^2$ . This can be achieved so that

$$\{\text{arcs } \alpha\} \cap \{\text{RIBBONS}\} = \emptyset.$$

Moreover, the various  $\pi \mid \alpha$  inject and are disjoint from each other. [In order to explain this, notice that any intersection  $\pi\alpha_1 \cap \pi\alpha_2$  would be a double point of  $\pi \mid \sum E_i^2$ . But these are all concentrated in the RIBBONS from which our arcs  $\alpha$  are disjointed, by construction.]

We have not tried to draw here the RIBBONS too, so as not to overcharge the figure. But the important point is that the RIBBONS in question do not separate the  $t$ 's from  $\partial E^2$ , allowing us to put in the  $\alpha$ 's.

Now, we join the holes  $t$  by arcs  $\alpha$ , to  $\partial E^2$ , **in an acyclic manner**, as it is suggested in the Figure 2; this is done without touching the RIBBONS of the immersion  $\sum_1^P E_i^2 \xrightarrow{\pi} \partial Y_{\text{small}}^4$ .

Next, we enlarge our manifold  $Y_{\text{small}}^4$  with small dilatations  $D^2(\alpha \cup \pi(\alpha))$ , each contained in a vertical plane, and which at the level of  $N^4 \left( Y^4 \cup \sum_1^P E_i^2 \right)$  (see (3.1)), means addition of 2-handles. These 2-handles, going along the linear chain above, **exactly cancel** the 1-handles from (3.1). So

$$N^4 \left( Y_{\text{small}}^4(\text{enlarged}) \cup \sum_1^P E_i^2 \right) \stackrel{\text{DIFF}}{=} N^4 \left( Y^4 \cup \sum_1^P D_i^2 \right),$$

meaning that the result of this construction, with  $Y^4(\text{enlarged}) = N^4 \left( Y^4 \cup \sum_{\alpha} D^2(\alpha \cup \pi(\alpha)) \right)$  is GSC.

Moreover, we have now a family of discs  $D^2 = D^2(\text{small}) \subsetneq \sum^2$  shaded in Figure 2, replacing the  $\sum_1^P E_i^2$  in (3.1) and such that

$$(3.2) \quad M_2 \left( \pi \mid \sum_1^P D_i^2(\text{small}) \right) = M_2 \left( \pi \mid \sum_1^P E_i^2 \right).$$

It is this family

$$(3.3) \quad \left( \sum_1^P D_i^2(\text{small}), \sum_1^P \partial D_i^2 \right) \subset (\partial Y_{\text{small}}^4(\text{enlarged}) \times [0, 1], \partial Y_{\text{small}}^4)$$

with which we will work from now on. That is what  $Y^4, D^2$ , will mean now.

Now, imagine for a split-second, that we would also know, from the very beginning that actually

$$M_2 \left( \pi \mid \sum_1^P D_i^2 \right) = \emptyset.$$

(This is, of course, a totally unrealistic assumption, at the present stage in the game, which we make here just for the sake of the argument, but let us still imagine ...)

Then, by a process of **raising the see level** inside the collar  $\partial Y_{\text{small}}^4(\text{enlarged}) \times [0, 1]$ , we could realize a diffeomorphism

$$N^4 \left( Y_{\text{small}}^4 \cup \sum_1^P D_i^2 \right) + \{3\text{-handles}\} \stackrel{\text{DIFF}}{=} Y_{\text{small}}^4,$$

[each 3-handle gotten by joining  $D_i^2$  to  $\pi D_i^2$  along the lines  $[y, \pi y]$ ,  $y \in D_i^2$ ].

This diffeomorphism would imply our desired result, but only under the outrageous assumption we just made. Going now back to real life and to (3.3), our RIBBONS are pairs of curves which inside  $M_2 \left( \pi \mid \sum_1^P D_i^2 \right) \subset \sum_1^P D_i^2$  (meaning from now the  $D_i^2(\text{small})$ ) look like in Figure 3. What we see there is a

$$\{\text{vertical plane}\} \subset \partial Y_{\text{small}}^4 \times [0, 1],$$

such that the  $\pi^{-1}\pi(\text{RIBBON})$ , lives (for every individual RIBBON) inside such a plane. Moreover, we have exactly

$$\pi^{-1}\pi(\text{RIBBON}) \subset \{\text{RECTANGLE}\} \subset \{\text{our vertical plane}\}$$

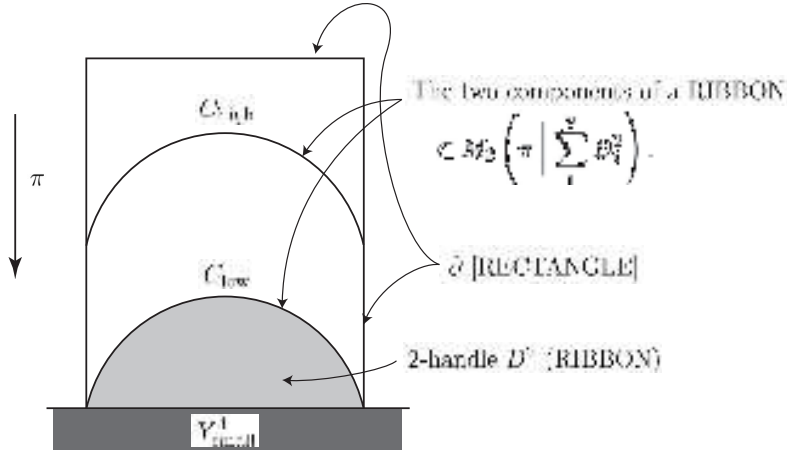


Figure 3.

The RIBBON, at the source, inside the collar  $\partial Y_{\text{small}}^4 \times [0, 1]$ , and its projection  $\pi$ . Here the RIBBON  $\equiv (C_{\text{high}}, C_{\text{low}})$ , with  $C_{\text{high}} \subset \text{int } D^2(\text{small})$  (3.3) and with a proper embedding  $(C_{\text{low}}, \partial C_{\text{low}}) \subset (D^2, \partial D^2)$ .

Notice now the following facts:

- a) Since the 2-handles  $D^2(\alpha \cup \pi(\alpha))$  have exactly cancelled the 1-handles in (3.1), we find that

$$Z^4 \equiv N^4 \left( Y_{\text{small}}^4 \cup \sum_1^P E_i^2(3.1) \right) \cup \sum_{\alpha} D^2(\alpha \cup \pi(\alpha)) \stackrel{\text{DIF}}{=} \text{(a diffeomorphism already mentioned)} \stackrel{\text{DIF}}{=} N^4 \left( Y_{\text{small}}^4(\text{enlarged}) \cup \sum_1^P D_i^2(\text{small})(3.3) \right) \in \text{GSC}.$$

(This really is a reminder.)

- b) To  $Z^4$  we may add the vertical 2-handles  $D^2(\text{RIBBON})$  which are shaded in the Figure 3 and then we get, as a consequence of a),  $V^4 \equiv Z^4 + \sum_{\text{RIBBONS}} D^2(\text{RIBBONS}) \in \text{GSC}$ .

- c) Consider now

$$W^4 \equiv N^4(Y_{\text{small}}^4(\text{original})) \cup \sum \{\text{all the vertical } 2^{\text{d}} \text{ smooth } \textit{dilatation} D^2(\alpha \cup \pi(\infty)) \text{ and } D^2(\text{RIBBON})\}.$$

We clearly have  $W^4 \stackrel{\text{DIF}}{=} Y_{\text{small}}^4(\text{original})$ .

- d) The  $C_{\text{low}}$ 's in Figure 3, breaks the system  $\sum_1^P D_i^2$  (3.3) into a larger system of even smaller discs, call it  $\sum_1^P D_i^2$  with  $Q > P$  (but we will not change the notation for the  $D_i^2$ 's). This comes with an embedding

$$\begin{array}{ccc} \left( \sum_1^Q D_i^2, \sum_1^Q \partial D_i^2 \right) & \hookrightarrow & (\partial W^4 \times [0, 1], \partial W^4 \times \{0\}) \\ & & \downarrow \pi \\ & & \partial W^4 \end{array}$$

which is now such that  $M_2 \left( \pi \left| \sum_1^Q D_i^2 \right. \right) = \emptyset$ . Moreover, clearly also

$$W^4 \cup \sum_1^Q D_i^2 \stackrel{\text{DIF}}{=} Z^4 + \sum_{\text{RIBBON}} \{\text{the 2-handles } D^2(\text{RIBBON})\} \in \text{GSC}.$$

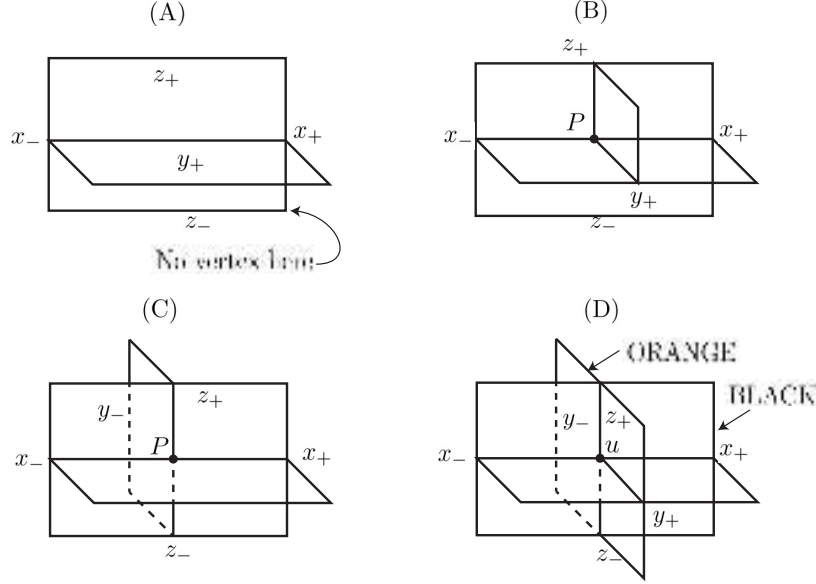
Applying once more the argument of the raising of sea-level, we find that  $W^4$  itself is GSC and this proves our lemma. It should be noticed that it is essential for our whole argument that the projections  $\partial Y_{\text{small}}^4 \times [0, 1] \xrightarrow{\pi} \partial Y_{\text{small}}^4$  and  $\partial W^4 \times [0, 1] \xrightarrow{\pi} \partial W^4$  are compatible; there is only one notion of VERTICALITY which is involved in both.  $\square$

Once Lemma 3 has been proved, we move back to our main theme. The important fact, at this point, is that in the context of (2) we are presented with two infinite collapses, or collapsing flows.

(4.A) As a simple consequence of the way in which  $\Delta_1^4$  is defined, we have a RED collapse  $\text{int } \Delta_1^4 \searrow \Delta_{\text{small}}^4$  (actually a compact  $\Delta_1^4 \searrow \Delta_{\text{small}}^4$  too, but that one we will never use).



(4.B) From Barry's result  $X^4 = \text{int } \Delta_1^4 \underset{\text{DIFF}}{=} R^4$  (see (1)), follows the existence of a second, BLUE collapse  $X^4 \searrow \text{pt.}$



**Figure 4.**

The local models for  $X^3(\text{GPS})$ . Any permutation of  $(x, y, z)$  or of signs, is of course OK too. The (B), (C) are, of course, homeomorphic, but NOT affinely so. So, for us they count as distinct. Of course also, the BLACK (A), (B), (C) may be orange, while in (D) we can also permute BLACK  $\Leftrightarrow$  ORANGE.

The only 1-skeleton present in (D) is  $[x_-, x_+] \subset X^1(\text{BLACK})$ , and this is like in the Figure 4-(A). I mean here the only 1-skeleton of one of the  $X^3(\text{COLOURS})$ 's. But then, when we move to  $X^4$  then all the lines in our figure will be in the  $X^1 \equiv \{1\text{-skeleton of } X^4\}$ .

Remember that BLACK and ORANGE play a symmetrical role in all our story.

Both of these collapsing flows will be used and one of the difficulties which we will have to overcome is that, a priori they cut transversally through each other in an uncontrollable fashion, and with a messy situation

$$(\text{RED collapsing flow}) \pitchfork (\text{BLUE collapsing flow}),$$

with which we could not live.

Our  $\Delta^4$  in (2) will be a subcomplex of a specially chosen cell-decomposition of  $R^4 = X^4$ , something which we will introduce very explicitly, next. We will call it a **“general parallelepipedic structure”** of  $X^4 = R^4 = R^3 \times R = X^3 \times R$ , which I will currently call G.P.S. structure (sorry for the possible confusion!).

In what follows next, we will work with  $R^4$  coming with its affine structure and with the coordinate system  $(x, y, z, t)$ ; occasionally, the euclidean metric will also be invoked.

The time axis  $R$  will be equipped with the lattice  $Z \subset R$ , the points of which will be labelled  $\dots < t_{-1} < t_0 < t_1 < t_2 < \dots$  and we will consider the spatial slices  $X^3 \times t_i = R^3$ . Here comes now the

**Definition 5.** We will define the **GPS** structures, first for  $R^3 = X^2 \times t_i$  and then for  $R^4 = R^3 \times R$ . Here is the list of the GPS features.

- a) Each  $X^3 \times t_i$  comes with two cell-decompositions  $X^3(\text{BLACK})$  and  $X^3(\text{ORANGE})$ , which are  $t_i$ -independent. The  $X^\varepsilon(\text{COLOUR})$  will be the  $\varepsilon$ -skeleton. We will ask that  $X^2(\text{BLACK})$  and  $X^2(\text{ORANGE})$  should cut transversally through each other, so that

$$X^1(\text{BLACK}) \cap X^1(\text{ORANGE}) = \emptyset \subset X^3 \times t_i.$$

- b) The local models for the  $X^2(\text{COLOUR})$  should be **generic**, like in Figure 4, where the  $X^2(\text{COLOUR}) \subset X^3 \times t_i$  are, locally, displayed. What we see in the Figure 4-(D) is the local model for the intersection  $X^2(\text{COLOUR}) \cap X^2(\text{DUAL COLOUR}) \subset X^3 \times t_i$ . Then, the point  $u$  is NOT a vertex of any of the two individual  $X^2(\text{COLOUR})$ , but certainly a vertex of the GPS structure of  $X^3 \times t_i$ , for which the cell-decomposition is canonically defined, starting from  $X^2(\text{BLACK}) \cup X^2(\text{ORANGE})$ .
- c) We move now to  $R^4$  which, when endowed with its GPS-structure, will be denoted  $X^4$ , of 2-skeleton  $X^2(\text{GPS})$ . We want that, after the interiors of some 2-cells, generically called  $D^2(\gamma^1) \subset X^2(\text{GPS})$ , are deleted, we should have a collapse

$$X^2(\text{GPS}) - \bigcup \text{int } D^2(\gamma^1) \searrow \text{pt.}$$

[Morally speaking the  $D^2(\gamma^1)$ 's are to be killed by a 3<sup>d</sup> collapse, but this, so called “BLUE” 3<sup>d</sup> collapse, will never be used. It ain't there.]

- d) Inside each  $X^3 \times t_i$  the 2<sup>d</sup> collapse from c) above should have **purely linear trajectories, without bifurcation**, like the blue arrow in Figure 5 may suggest.
- e) When we move from  $X^3 \times t_i$  to  $X^4$ , then each vertex  $P \in X^3 \times t_i$  carries at most one of the edges  $P \times [t_i, t_{i+1}]$  OR  $P \times [t_i, t_{i-1}]$  and NEVER BOTH. But also, sometimes there are no temporal edges.
- f) The  $X^4$  has no vertices outside of the  $X^3 \times t_i$ 's.

[Implicit in this definition, where  $P$  is a vertex of  $X^4$  the edges coming out of  $P$  can only go in one of the directions  $\pm x, \pm y, \pm z$  or  $\pm t$  and not all of these choices are realized, for any given vertex  $P$ .] We will denote by  $X^4$  the generic GPS subdivision of  $R^4$  and by  $X^\varepsilon$  its  $\varepsilon$ -skeleton.

We think from now on in terms of  $X^4 = (R^4, \text{ with a GPS-structure})$ . Of course there is here a potential source of confusion since  $X^3(\text{GPS})$  has already been used for  $(R^3, \text{ with a GPS-structure})$ . When this confusion is not resolved by the context, we may write  $X^{(3)}$  for the 3-skeleton of  $X^4$ . End of (5).

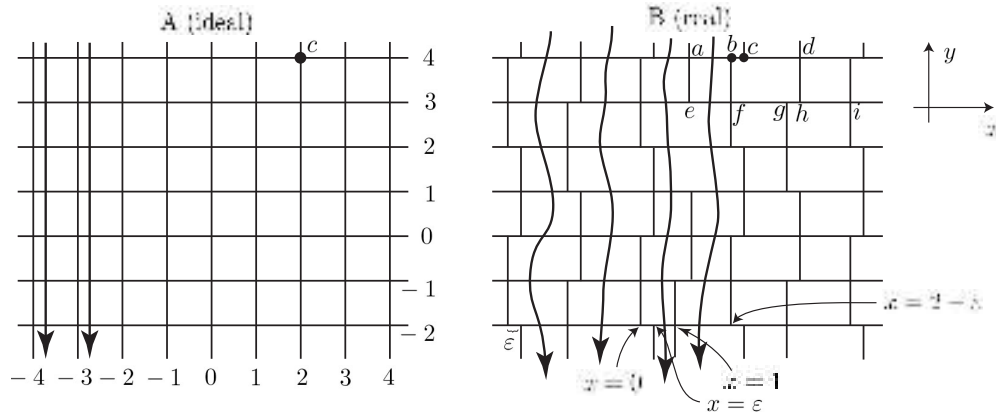


Figure 5.

The  $X^1(\text{COLOUR})$  in  $R^2(x, y)$ , in ideal and in real life. The BLUE arrows stand for the collapsing flow of  $R^2$ . In A (ideal) vertical lines are black and horizontal ones orange. This goes over to the deformed version B (real).

Of course, the GPS structure is not uniquely determined by the definition above, and many choices are possible.

So I will describe now the STANDARD GPS STRUCTURE.

We will start with  $R^3$  and, ideally, the  $X^2(\text{COLOUR})$ 's should be two perfectly cubical structures cutting transversally through each other, like Figure 5-(A) may suggest, with one dimension less. But then, this does not satisfy the genericity condition in (5)-b). Let us say, more precisely, that in the ideal case we chose, for  $(m, n, p) \in Z$ ,  $X^2(\text{BLACK})(\text{ideal}) = (x = 2m, y = 2n, z = 2p)$ ,  $X^2(\text{ORANGE}) = (x = 2m + 1, y = 2n + 1, z = 2p + 1)$ . Then we go to the real life situation, in two steps.

I) We consider first, in  $R^2(x, y)$ , the  $X^1(\text{BLACK})(\text{ideal})$ ,  $X^1(\text{ORANGE})(\text{ideal})$ , like in the Figure 5-(A). Next, we perturb this, generically, as follows:

The lines  $y = n \in Z$ , independently of their colour, are left put and the black line  $x = 2m$  is broken into alternating successive pieces  $x = 2m$ , and  $x = 2m + \varepsilon$ , with the pattern suggested by Figure 5-(B); something similar if then done for the orange lines  $x = 2m + 1$ , and this is also represented in Figure 5-(B). After this we have an inclusion

(\*)  $\{\text{the bicoloured structure in Figure 5-(B)}\} \times \{-\infty < z < +\infty\} \subset \{2\text{-skeleton of the } X^3 \times t_i\}$ . But the 2-skeleton in question contains some horizontal,  $z = \text{const}$ , pieces too.

With the structure from Figure 5-(B), we consider all the little edges from the figure in question, like for instance  $[a, b]$ ,  $[b, c]$ ,  $[c, d]$ ,  $[a, e]$ ,  $[b, f]$ ,  $[e, f]$ ,  $[d, h]$ , and all the thin vertical bands  $\{\text{[little edge]} \times (-\infty < z < +\infty)\}$ , appropriately divided into little squares by the next step II) are the 2<sup>d</sup> BLUE collapsing flow lines. These occur in (\*), and see here also Figure 35.2. The BLUE 2<sup>d</sup> collapsing flow goes in the direction  $\pm z$  (see Figure 35.2).

**Remarks.** A) The BLUE flow lines in Figure 5, living horizontally in  $(x, y)$  are *not* our real-life BLUE 2<sup>d</sup> collapsing flow. They just explain what we mean by linear trajectories in (5)-d. The real life 2<sup>d</sup> BLUE flow-lines are even straighter than in the Figure 5-(B).

II) If we take the whole  $z$ -flow of the ORANGE (respectively BLACK) structure in Figure 5-(B) and (\*), and then cut it with ORANGE (respectively BLACK) horizontal planes, namely ORANGE planes  $z = 2p + 1$  and BLACK planes  $z = 2p$ , the condition b) is still not verified at the intersections of these horizontal planes with the vertical lines going through the vertices from Figure 5-(B). We will name  $X^1(\text{BLACK/ORANGE})$  the 1<sup>d</sup> structure from Figure 5-(B). It has three kind of vertices: pure BLACK vertices, like  $b, c$ , pure ORANGE vertices, like  $g, h$  and then mixed vertices like  $a, f$  or  $d$ . Consider now the BLACK vertex  $c$  from Figure 5-(B) and the four 1/4 planes of BLACK colour, at  $z = 2p$ , neighbouring it in the ideal case. In the real life case, at  $((b), c)$  we perturb the four 1/4-planes, as follows:  $(x > 0, y > 0) \Rightarrow z = 2p$ ,  $(x > 0, y < 0) \Rightarrow z = 2p + \varepsilon$ ,  $(x < 0, y < 0) \Rightarrow z = 2p + 2\varepsilon$ ,  $(x < 0, y > 0) \Rightarrow z = 2p + 3\varepsilon$ . We do a similar change at the ideal ORANGE vertices (with  $z = 2p$  replaced by  $z = 2p + 1$ ), i.e. this time we handle  $(g, h)$ . The BLACK modification renders generic  $a$  and also  $d$ , and the ORANGE modification takes care of  $i$  and  $f$ . Here one uses the fact that  $a, b$  are placed between two consecutive black vertices on the same horizontal and then, similarly  $f, i$  between two consecutive orange vertices, again in the same horizontal.

So, with all these things we have by now, at level  $X^3 \times t_j$  the correct structures  $X^2(\text{BLACK})$  and  $X^2(\text{ORANGE})$ . Their union, i.e. the (\*) above, to which the horizontal BLACK and ORANGE little squares are added *is* the  $X^2(\text{GPS}) \times t_j$ . But before we can move to 4<sup>d</sup>, we will need a refinement of the  $X^3(\text{GPS})$  we have just introduced.

The  $X^1(\text{BLACK/ORANGE})$  in Figure 5-(B) is clearly a  $(Z + Z)$ -tessellation of  $R^2$  and so are the  $X^1(\text{BLACK})$ ,  $X^1(\text{ORANGE})$ , independently. Similarly,  $X^3(\text{GPS}) \approx X^2(\text{BLACK/ORANGE})$ ,  $X^2(\text{BLACK})$ ,  $X^2(\text{ORANGE})$  are each of them a  $(Z + Z + Z)$ -tessellation of  $R^3$ . The  $X^2(\text{BLACK/ORANGE})$  is a symmetric  $\varepsilon$ -generic deformation of the standard integral cubical subdivision of  $R^3$ . We need to identify now, inside  $X^2(\text{BLACK/ORANGE})$  two coarser tessellation (coarser meaning with larger fundamental domain) call them  $2X^2(\text{COLOUR})$ , with the feature that,  $2X^2(\text{ORANGE})$  and  $2X^2(\text{BLACK})$ , i.e.  $2X^2(\text{ORANGE}) \subset X^2(\text{ORANGE}) \subset X^2(\text{BLACK/ORANGE}) \supset X^2(\text{BLACK}) \supset 2X^2(\text{BLACK})$ , are transversal to each other. At the  $R^2$ -level of the ideal Figure 5-(A) we may take  $2X^1(\text{BLACK})(\text{ideal}) = \{x = 6m, y = 6n\}$ ,  $2X^1(\text{ORANGE})(\text{ideal}) = \{x = 6m + 3, y = 6n + 3\}$  and from there one continue as before. Importantly, we have  $2X^2(\text{BLACK}) \pitchfork 2X^2(\text{ORANGE})$  (i.e. transversal contact, with  $2X^1(\text{BLACK}) \cap 2X^1(\text{ORANGE}) = \emptyset$ , in  $X^3 \times t_i$ .)

We move now to the  $X^4$  and in order to define its standard  $4^d$  GPS structure, it is sufficient to specify the  $X^2$ , which afterwards can be filled in canonically (i.e. linearly) with  $3^d$  and  $4^d$  cells.

All the features in definition (5) will be satisfied, if we take

$$(6) \quad X^2 \equiv \sum_i \underbrace{X^2(\text{GPS}) \times t_i}_{\text{bicollared}} + \sum_i 2X^1(\text{BLACK}) \times [t_{2i-1}, t_{2i}] + \sum_i 2X^1(\text{ORANGE}) \times [t_{2i}, t_{2i+1}].$$

This ends the definition of our “standard” GPS structure which will be our normal way to put flesh and bones on the DEFINITION (5).

In our context,  $\Delta^4 = \Delta_{\text{small}}^4 \subset X^4$  is a subcomplex and so its 2-skeleton  $\Delta^2$  is a subcomplex of  $X^2$ . We will use the notation

$$\Gamma(1) \equiv \{1\text{-skeleton of } \Delta^4\} \subset \Gamma(\infty) = \{\text{the } X^1, 1\text{-skeleton of } X^2\}.$$

We will express now the basic RED and BLUE collapses from (4-A and B) by the following lemma, which is a detailed version of the (4).

**The P.L. Lemma 3.1.** *For  $R^4$  (with its DIFF standard structure), there is a GPS structure  $X^4$  such that*

- 1)  $\Delta_{\text{Schoenflies}}^4 \subset \Delta^4 \cup \partial\Delta^4 \times [0, 1) = R^4$  is a subcomplex of it. And  $X^4$  will have all our desirable BLUE collapsibility properties. Here  $\Delta_{\text{Schoenflies}}^4 = \Delta_{\text{small}}^4$ , of course.
- 2) The infinite complex  $X^4 - \text{int } \Delta^4$  has a RED collapse

$$X^4 - \text{int } \Delta^4 \searrow S^3 = \partial\Delta^4.$$

This last feature 2) will be called the PROPERTY (P). So the PL-lemma says that there is a GPS structure with property (P) or, said differently, we can realize both features (4-A), (4-B) with the same GPS structure. [Only the  $2^d$  part of the BLUE collapse will ever be needed.]

Now, what Barry’s classical result (0) tells us is that, with the standard affine (and hence DIFF too) structure of  $R^4 = \Delta^4 \cup \partial\Delta^4 \times [0, 1)$  there is a strictly cubical subdivision, s.t.  $\Delta_{\text{Schoenflies}}^4$  is a subcomplex. I am stressing here this “affine”, in view of what is coming next, but the euclidean metric of  $R^4$  will have to be, occasionally, invoked too.

**Sublemma 3.1-A.** *There is a cubical subdivision of  $R^4$ , like above, s.t. the  $R^4 - \text{int } \Delta_{\text{Schoenflies}}^4$  has the property (P).*

**Proof.** Here are the successive steps of the argument.

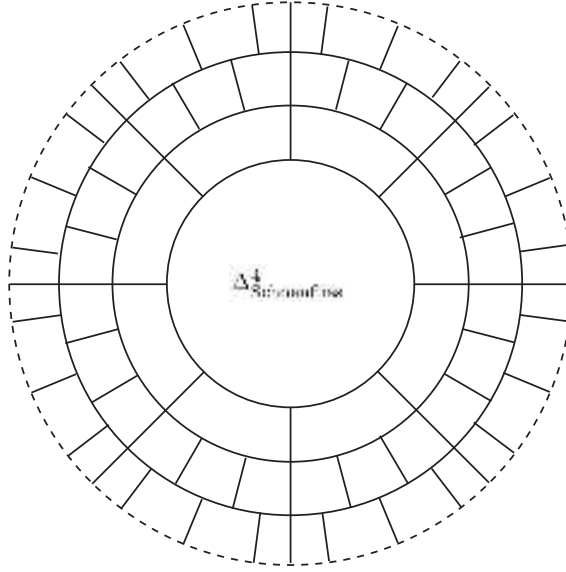
I) We reformulate explicitly the fact that  $R^4 - \text{int } \Delta^4$  has property (P). For the locally affine manifold  $R^4 - \text{int } \Delta^4$  there exists a triangulation  $T$  by convex affine simplexes, s.t.

(6.1)  $T$  has the property (P).

(6.2) There is a positive lower bound  $\varepsilon > 0$  s.t.

- i) For each simplex  $\sigma$  of  $T$ ,  $\|\sigma\| > \varepsilon$ ,
- ii) {the angles of the faces of  $\sigma$  with the coordinate  $1^d$ ,  $2^d$  or  $3^d$  planes defined by  $(x, y, z, t)$ }  $> \varepsilon$ .

Figure 5.1 should suggest the proof of (6.2). Of course, the full detailed argument here has to invoke the DIFF HAUPTVERMUTUNG of J.H.C. Whitehead, but this should be quite standard. There is no cubical structure at all involved in our step II, that will be coming next. Also what we mean by “cubical” is not just that the 3-cells are cubes, but that these cubes are nicely aligned with  $(x, y, z)$ .



**Figure 5.1.**

Idea for the proof of (6.2). The dotted line is the infinity of  $R^4$  and the plain circles are equidistant.

II) Let us denote by  $X$  the strictly cubical structure of  $R^4$  and by  $X'$  the cell-decomposition  $X \cap T$ , subdivision of  $T$ .

**Sub-Sublemma 3.1-B.** *When restricted to  $R^4 - \text{int } \Delta^4$ , the  $X \cap T$  continues to have Property (P).*

**Proof.** One gets  $X \cap T$  by **bisecting** the affine-convex simplexes of  $T$  by affine planes of dimension 1, 2, 3. There are the planes defined by  $X$ . Our desired conclusion follows easily from this fact. [Property (P) is invariant under barycentric subdivisions, stellar subdivisions and, more up to the point, under **Siebenmann's bisections**.]

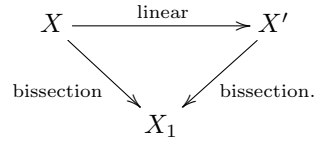
The “bisections” mentioned above are to be introduced formally in the next point III).

III) I remind the reader the basic facts of Larry Siebenmann's theory of bisections.

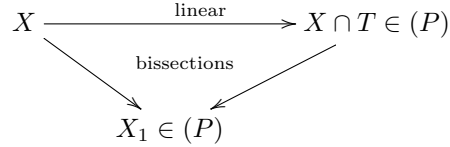
Siebenmann starts by introducing *cellulations*, which are an extension of simplicial complexes: instead of using simplexes we use now compact cells  $D$  with a linear-convex structure. The notion of (linear) subdivision extends in an obvious way to cellulations and, also, instead of subcomplexes we can introduce now sub-cellulations. What we have gained with this approach is, among other things, the following useful fact: if  $Y \subset Z$  is a sub-cellulation, then any subdivision  $Y'$  extends canonically to a subdivision of  $Z$ , not affecting the open cells in  $Z - Y$ . An important class of subdivisions are the **BISSECTIONS**. These are localized at the level of an  $i$ -cell  $D^i$  and are obtained by cutting  $D^i$  with a hyperplane  $H^{i-1} \subset D^i$  and splitting  $D^i$  itself and any sub-cell of  $D^i$  met by  $H^{i-1}$ , in the obvious way. Our “useful fact” above extends to bisections.

One should notice that no genericity conditions are required here for the position of the hyperplane  $H^{i-1} \subset D^i$ . As mentioned above, bisections, barycentric subdivisions or Alexander’s stellar subdivisions clearly preserve Property (P), but this kind of thing is a priori nor clear for general linear subdivision. But then here comes Siebenmann’s very useful version of Alexander’s old classical lemma.

**The Siebenmann Lemma.** *Let  $X$  be a cellulation and  $X \rightarrow X'$  a linear subdivision of  $X$ . There exists then a cellulation  $X_1$  such that one can go both from  $X$  and from  $X'$  to  $X_1$ , via bisections*



Now,  $X \rightarrow X \cap T$  *is* a linear subdivision of  $X$  too, and so one can apply Siebenmann’s lemma and get a common bisection  $X_1$ . Since  $X \cap T \in (P)$  and  $X \cap T \rightarrow X_1$  is a bisection, we also have  $X_1 \in (P)$ . Diagrammatically we have here



IV) We consider the full  $X$  for  $R^4$  with its  $X \cap T \mid (X^4 - \text{int } \Delta^4) \in (P)$  and the bisection  $X \rightarrow X_1$ , considered for the full  $X$ . (Unlike what happens for, let us say, barycentric subdivisions, a bisection of a subcomplex is, automatically, a bisection of the whole complex.)

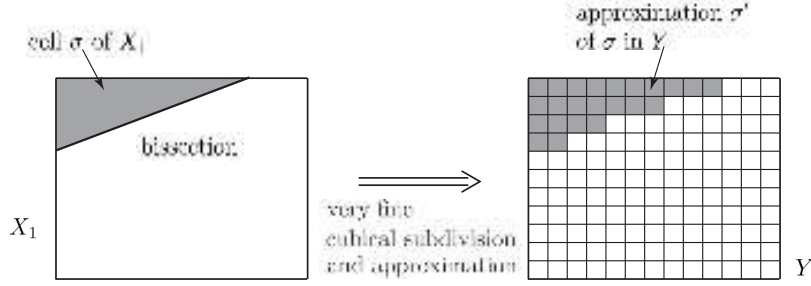
From (6.2) it follows that there is a uniform, very fine cubical subdivision of our cubical  $X$ , call it

$$X \xrightarrow{\text{very fine cubical subdivision}} Y,$$

s.t. inside  $Y$  we can MIMICK the bisection  $X \rightarrow X_1$  by subcomplexes of  $Y$  which are very close approximations of convex hyperplanes. The idea of this, is suggested in the Figure 5.2 below.

When each cell  $\sigma$  of  $X_1$  is replaced by  $\sigma' \subset Y$  we get a cell-decomposition  $X''$  of  $R^4$ , which is very close to  $\sigma$  and such that:

- a) The cells  $\sigma'$  of  $X''$  are nearly-convex.
- b)  $X'' \in (P)$ .



**Figure 5.2.**

MIMICKING

Consider now the subdivision  $X'' \rightarrow Y$  which is gotten by cutting the  $\sigma$ 's with the hyperplanes suggested in the RHS of Figure 5.2. Because of a) above, for the same reasons as in the proof of the Sublemma 3.1-(B) we have the implication

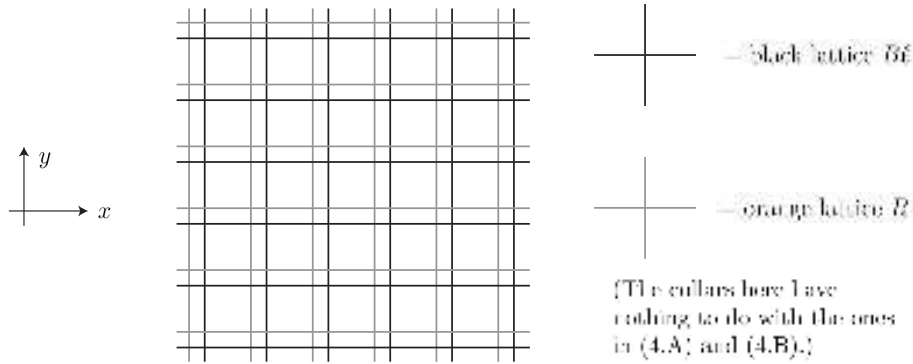
$$X'' \in (P) \implies Y \in (P).$$

This proves our Sublemma 3.1-A.  $\square$

In order to clinch the proof of the PL Lemma 3.1 we have to show how to go from cubical subdivisions to GPS structures without losing neither the BLUE  $2^d$  collapsibility nor the RED property (P).

We start with the cubical subdivision  $Y$  which has property (P) and which hence satisfied the Sublemma 3.1-A.

**Sublemma 3.1-C.** *Via appropriate slidings and appropriate subdivisions, none of which violate, neither the BLUE  $2^d$  collapsibility nor the RED property (P), we can change  $Y$  into the standard GPS structure:*



**Figure 5.3.**

This figure goes with the explanations for DILATATING SLIDINGS.

**Proof.** All our manipulations now will be  $2^d$  and the extension to  $3^d$  and  $4^d$  will be canonically automatic. It is well-known that, generically speaking,  $2^d$  sliding moves of J.H.C. Whitehead violates collapsibility. But there is a category of sliding moves, which I will call **dilatating slidings** which do **not** violates collapsibility.

For these special slidings there is no disconnecting of smooth strata of maximal dimension. Here is our typical paradigmatic example.

Consider, to begin with the following  $2^d$  infinite complex  $K$ .

$K \equiv \{\text{the plane } z = 0 \text{ from Figure 5.3}\} \cup \{\text{for each line } L \text{ of the Black lattice } B\ell \text{ we consider the plane } L \times (-\infty < z < \infty), \text{ which we add to } (z = 0)\}.$

This obviously has an infinite collapse  $K \searrow \text{pt}$ . Move now from  $K$  to the following

$K' \equiv \{\text{the same } z = 0 \text{ as above}\} \cup \{\text{for each line } L \in B\ell \text{ we add the } \frac{1}{2}\text{-plane } L \times [-\infty < z \leq 0]\} \cup \{\text{for each line } \Lambda \in R \text{ we add the } \frac{1}{2}\text{-plane } \Lambda \times [0 \leq z < \infty]\}.$

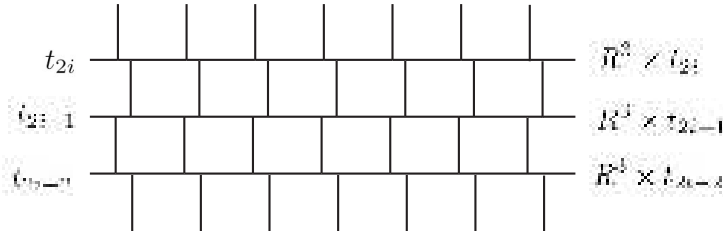
The  $K \Rightarrow K'$  is our typical dilatating slide and clearly  $K' \searrow \text{pt}$  too. There are of course much simpler and more simple-minded examples.

We look now at the  $2^d$  skeleton of  $Y$ , call it  $Y^2$ , and which has the following obvious structure. We have a very fine cubical subdivision of the time axis  $t \in R$ , call it

$$\dots < t_{-1} < t_0 < t_1 < t_2 < \dots$$

Then, there is a time independent cubical subdivision  $Z^2$  of  $R^3 = (x, y, z)$  with 1-skeleton  $Z^1$ . With this, we have the following structure for our  $Y^2$

$$Y^2 = \left( \sum_i Z^2 \times t_i \right) \cup (Z^1 \times (-\infty < t < \infty)).$$



**Figure 5.4.**

The vertical arcs of the form  $[t_{2j-1}, t_{2j}]$  are here ORANGE.

We proceed now with the dilatating sliding suggested in Figure 5.4. This implements the e) from (5), and also f) from the same (5). On each individual  $R^3 \times t_j$  we have a strictly cubical subdivision. From this structure one can then go to the standard GPS structure by more cubical subdivisions on the  $R^3 \times t_j$ 's, followed by small slidings of the dilatating type.

This ends the proof of our PL Lemma 3.1.  $\square$

We work now with the structure produced by the PL Lemma 3.1.

**Lemma 4.** 1) Inside  $\Gamma(\infty)$  we will consider “*spots*”, small open arcs far from any vertex. There will be two sets of them

$$(7) \quad R(\text{for RED}) \subset \Gamma(\infty) \supset B(\text{for BLUE}),$$

and we should also think of them as 1-handles. A given edge  $e \in \Gamma(\infty)$  will carry at most one of the  $r_i \in R$  or  $b_j \in B$ . Also, if by any chance we find  $r_i \in e \ni b_j$ , same  $e$ , then  $r_i = b_j$ , and this is how  $R \cap B$  is generated.



We will introduce the more appropriate notations

$$(8) \quad R \cap \Gamma(1) = \{R_1, R_2, \dots, R_n\}, \quad R - \{R_1, R_2, \dots, R_n\} = \{h_1, h_2, \dots\}.$$

The graphs  $\Gamma(1) - R \cap \Gamma(1)$ ,  $\Gamma(\infty) - R$ ,  $\Gamma(\infty) - B$  are trees but, generically, the  $\Gamma(1) - B$  is a disconnected union of **several** trees.

2) The  $X^2$  will be gotten by adding 2-cells along a  $\{\text{link}\} \subset \Gamma(\infty)$ . We will have two disjoint partitions for this link, namely

$$(9) \quad \begin{aligned} \{\text{link}\} &= \sum_1^{\bar{n}} \Gamma_i + \sum_1^{\infty} C_j + \sum_1^{\infty} \gamma_k^0 \text{ (RED partition)} \\ &= \sum_1^{\infty} \eta_\ell + \sum_1^{\infty} \gamma_m^1 \text{ (BLUE partition)} \end{aligned}$$

where, to begin with,  $\sum_1^{\bar{n}} \Gamma_i \subset \Gamma(1)$  and the 2-skeleton of  $\Delta_{\text{Schoenflies}}^4$  is here

$$\Delta^2 = \Gamma(1) + \sum_1^{\bar{n}} D^2(\Gamma_i);$$

next,

$$\begin{aligned} X^2 &= \Gamma(\infty) \cup \sum_1^{\bar{n}} D^2(\Gamma_i) \cup \sum_1^{\infty} D^2(C_j) \cup \sum_1^{\infty} D^2(\gamma_k^0) = \\ &= \Gamma(\infty) \cup \sum_1^{\infty} D^2(\eta_\ell) \cup \sum_1^{\infty} D^2(\gamma_m^1). \end{aligned}$$

Here  $D^2(\text{curve})$  is the 2-cell attached along the respective curve. Moreover, clearly  $\bar{n} = \# D^2(\Gamma) \geq n = \# R \cap \Gamma(1)$ ,  $\# B \cap \Gamma(1) \geq n$ .

The  $D^2(\gamma^0)$  (respectively  $D^2(\gamma^1)$ ) are exactly the 2-cells which are killed by the 3<sup>d</sup> part of the RED (respectively BLUE collapse), in (4). For the time being, at least, that is all we will say concerning the 3<sup>d</sup> collapses.

We have

$$X^2 = \Gamma(\infty) \cup \sum_1^{\bar{n}} D^2(\Gamma_j) + \sum_1^{\infty} D^2(C_i) + \sum_1^{\infty} D^2(\gamma_k^0) \supset X_0^2 \equiv \Gamma(\infty) \cup \sum_1^{\bar{n}} D^2(\Gamma_j) + \sum_1^{\infty} D^2(C_i).$$

Actually, we will leave alive in  $X_0^2$  a thin boundary collar, for each of the deleted  $D^2(\gamma_k^0)$ 's.

3) The 2<sup>d</sup> part of our RED and BLUE collapsing flows are expressed by the following two geometric intersection matrices

$$(10) \quad \begin{aligned} C_j \cdot h_p &= \delta_{jp} + \xi_{jp}^0, \text{ when } \xi_{jp}^0 \neq 0 \text{ implies } j > p, \\ \eta_\ell \cdot b_q &= \delta_{\ell q} + \zeta_{\ell q}^0, \text{ when } \zeta_{\ell q}^0 \neq 0 \text{ implies } \ell > q. \end{aligned}$$

End of Lemma.

[An important digression. The kind of infinite matrices which occur in (10) will be called of the “**easy id** + **nilpotent**” type. The easy id + nil implies GSC, as we will explain more in detail soon. But then, there

is also the “id + nilpotent of the difficult type”, with the final inequalities occurring in (10) reversed. This occurs, for instance, for the classical Whitehead manifold  $\text{Wh}^3$ , which certainly is **not** GSC. [It is not GSC for many reasons, the first one coming to my mind being that for open 3-manifolds, GSC implies  $\pi_1^\infty = 0$ .] Let us be a bit more precise concerning our condition of easy id + nilpotent. To simplify the exposition, I will concentrate on  $\eta_\ell \cdot b_q$ , when  $\Delta^2$  does not play any special role, but, with appropriate changes similar things can be said for  $C_j \cdot h_p$  too.

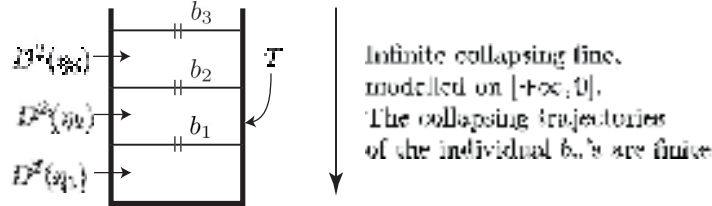
To begin with, our indices belong to an appropriate ordered set, NOT necessarily totally ordered, and with  $\zeta_{\ell q}^0 \neq$  in (10) implying  $\ell > q$ , satisfying also the following additional condition: From any element in our set of indices starts at least one back-going trajectory which is INFINITE. Trajectories mean things like  $\ell \rightarrow q$ , in the context above. Anyway, for the open manifolds concerned right now, this condition is natural.

Now, we are in a countable context for the  $\eta_\ell \cdot b_q$  and so, we can re-index things compatibly with  $\eta_\ell \cdot b_q =$  easy id + nil, and make our set of indices be  $Z_+$ . [We have a monomorphism

$$\{\text{our not necessarily totally ordered, but still ordered set}\} \rightarrow Z_+]$$

Our geometric intersection matrix concerns the 2-skeleton  $X^2$  of our  $X^4 = R^4$  and, with 1-skeleton  $X^1 \subset X^2$ , we have a tree  $T$  to which the  $h_q$ ’s and  $D^2(\eta_\ell)$ ’s get attached, so as to get an  $X^2 - \Sigma \text{int } D^2(\gamma^1) \subset X^2$ . With this, what  $C \cdot h =$  easy id + nilpotent means, is that one can go from  $T$  to the  $X^2 - \sum_1^\infty \overset{\circ}{D}^2(\gamma^1)$ , by an infinite sequence of dilatation. Alternatively, this can be also expressed in one of the following ways:

- There is an “infinite collapse”  $X^2 - \sum_1^\infty \overset{\circ}{D}^2(\gamma^1) \searrow T$ , OR
- (The GSC condition) The 1-handles cancel with the 2-handles in the manner suggested by the drawing below.



End of digression.]

In the digression above, we have worked with  $X^2$  (– the  $D^2(\gamma^1)$ ’s), but we will also need

$$(11) \quad X_0^2 \equiv \Gamma(\infty) \cup \sum_1^{\bar{n}} D^2(\Gamma_j) \cup \sum_1^\infty D^2(C_j) \subset X^2 \quad (X_0^2 = \{X^2 \text{ with the } D^2(\gamma^0)\text{'s deleted}\}),$$

which comes with a RED  $2^d$ -collapse  $X_0^2 \searrow \Delta^2$ . But then  $X_0^2$  is clearly limping BLUE-wise; there the full  $X^2$  is needed, and then the  $\gamma^1$ ’s get deleted, ...

We will need to extend this little theory, from  $X^2$  to two successive, more elaborate contexts, the  $X^2[\text{new}]$  and the  $2X^2$ . So, we will call, from now on, “old”, the context of Lemma 4, the role of which was to be a pedagogical introduction. But then, our real-life contexts will be, successively, the NEW context, and then the “DOUBLED” one. Here is what these will do for us.

On the one hand, they will put our  $\Delta^2$  in a protected position with respect to the BLUE flow. Then, they will eliminate the horrible complications stemming from the intersection of the two collapsing flows

$$(\text{RED } 2^d \text{ collapsing flow}) \pitchfork (\text{BLUE } 2^d \text{ collapsing flow}) = ?$$

Each of the individual flows is devoid of closed oriented orbits, but these may happily occur when we put them together. And, it is so that we could not live with such a thing.

**A first change of viewpoint, old  $\Rightarrow$  new.** The  $X^2$  from Lemma 4 is called  $X^2(\text{old})$  from now on.

And then, for the time being purely abstractly, we will introduce a fifth coordinate called  $\xi_0$ , in addition to  $(x, y, z, t)$ . We define

$$(12) \quad X_0^2[\text{new}] \equiv \left( X_0^2(\text{old}) - \sum_1^{\bar{n}} \overset{\circ}{D}^2(\Gamma_j) \times (\xi_0 = 0) \right) \underbrace{\cup}_{\Gamma(1) \times (\xi_0 = 0)} [\Gamma(1) \times [0 \geq \xi_0 \geq -1]]$$

$$\underbrace{\cup}_{\Gamma(1) \times (\xi_0 = -1)} \Delta^2 \times (\xi_0 = -1).$$

Forgetting for the time being about the BLUE collapse and keeping only the RED  $2^d$  collapsing flow alive, we have here a (RED)  $2^d$  collapse

$$(12.0) \quad X_0^2[\text{new}] \searrow \Delta^2 \times (\xi_0 = -1).$$

This should be obvious, but we will write explicitly below the relevant geometric intersection matrices. Here is now OUR CHANGE OF VIEWPOINT: We decree, from now on, that the  $\Delta^2 \times (\xi_0 = -1)$  **is** our  $\Delta^2$  of interest, its 2-cells **are** the  $D^2(\Gamma_i)$ ; the 2-cells of  $\Delta^2 \times (\xi_0 = 0)$  (which are anyway absent in  $X_0^2[\text{new}]$  and which will afterwards reappear in  $X^2[\text{new}] \supset X_0^2[\text{new}]$ , see (12.1) below), become new  $D^2(\gamma^0)$ 's and the new 2-cells involving an edge  $[0 \geq \xi_0 \geq -1]$  become new  $D^2(C_j)$ 's to be added to the previous  $D^2(C_j) \subset X_0^2(\text{old})$ . When we talk about the  $X_0^2[\text{new}]$ , keep in mind that both the  $\overset{\circ}{D}^2(\gamma^0) \subset X^2(\text{old})$  AND the  $D^2(\Gamma_i) \times (\xi_0 = 0) \subset \Delta^2 \times (\xi_0 = 0)$ , are absent, replaced by boundary collars.

Together, they will be the NEW family  $D^2(\gamma^0)$  of  $X_0^2[\text{new}]$ . Explicitly,

$$D^2(\gamma^0)[\text{new}] = \{\text{The } D^2(\Gamma_i) \times (\xi_0 = 0)\} + \{\text{The } D^2(\gamma^0)(\text{old})\}.$$

The sites  $R_i \times (\xi_0 = -1)$  for  $i = 1, 2, \dots, n$  are promoted as NEW  $R_i \subset \Delta^2 = \Delta^2 \times (\xi_0 = -1)$  and they live in  $\Gamma(1) \times (\xi_0 = -1)$ , of course. The old  $h$ 's stay put and, additionally to them, **every** edge  $e \subset \Gamma(1) \times (\xi_0 = 0)$  acquires an  $h$ , out of the blue. [The old  $R^i \times (\xi_0 = 0)$  are thereby all retrogrades as  $h$ 's.] BUT there is no  $h$  on the new edges  $P \times [0 \geq \xi_0 \geq -1]$ ,  $P \in \Gamma_0(1) \equiv 0$ -skeleton of  $\Delta^2$ . The RED collapse of  $X_0^2[\text{new}]$  proceeds as follows: We first collapse, normally  $X_0^2[\text{new}] \supset X_0^2(\text{old}) \searrow \{\Delta^2 \times (\xi_0 = 0)\}$  with the interiors of its 2-cells all deleted, they are now  $D^2(\gamma^0)$ 's, then starting from  $\Gamma(1) \times (\xi_0 = 0)$  we erase  $\Gamma(1) \times [0 \geq \xi_0 > -1]$  and we are left with our

$$\Delta^2 \equiv \Delta^2 \times (\xi_0 = -1), \text{ as we should.}$$

The RED story for  $X_0^2[\text{new}]$ , as told above, is OK, but since things are limping BLUE-wise, we introduce the following space, on the lines of (12) above

$$(12.1) \quad X^2[\text{new}] = X^2(\text{old}) \cup [\Gamma(1) \times [0 \geq \xi_0 \geq -1]] \cup \Delta^2 \times (\xi_0 = -1).$$

Notice that each  $e_i \subset \Gamma(1) \times (\xi_0 = 0)$  carries now an  $h_i = h(e_i)$  its RED dual 2-cell is  $D^2(C_i) = e_i \times [0 \geq \xi_0 \geq -1]$ . Similarly, the BLUE context of  $X^2[\text{new}]$ , each  $e_i \subset \Gamma(1) \times (\xi_0 = -1)$  carries a  $b(e_i)$  with dual the

$$D^2(\eta(e_i)) = e_i \times [-1 \leq \xi_0 \leq 0].$$

These things generate the little schematical figure below.



Figure 5.bis

The BLUE and RED collapsing flows inside

$$\Gamma(1) \times (0 \geq \xi_0 \geq -1) \subset X_0^2[\text{new}] \subset X^2[\text{new}].$$

Something should be stressed at this point. In the context of the collapse from (4-R, B) we had 2-cells  $D^2(\gamma^0)$ , respectively  $D^2(\gamma^1)$  killed by the corresponding 3<sup>d</sup> collapses. In our present, purely 2<sup>d</sup> abstract context, before the RED (respectively the BLUE) 2<sup>d</sup> collapses, the  $D^2(\gamma^0)$ 's (respectively the  $D^2(\gamma^1)$ 's) have to be deleted, **by decree** (not by 3<sup>d</sup> collapse, there ain't any, at least not in the BLUE context, when our game will always be purely 2-dimensional). Of course all our 2<sup>d</sup> objects will eventually be thickened into 4<sup>d</sup> manifolds, but we will not talk about that now.

For expository purposes, a RED 3<sup>d</sup> collapse will be introduced below, and very transiently only, in the NEW context. Afterwards, a ghostly memory of it will survive in our really serious, DOUBLED context. But never from now on will then be any trace of any 3<sup>d</sup> BLUE collapse. Now, we define for the  $X^2[\text{new}]$  in (12.1), the family

$$(12.2) \quad D^2(\gamma^1)[\text{NEW}] = \{\text{The } D^2(\Gamma_i) \times (\xi_0 = -1)\} + \{\text{The } D^2(\gamma^1)(\text{old})\}.$$

With this, we also introduce, for the same  $X_0^2[\text{new}]$ , the family

$$B[\text{new}] \equiv B(\text{old}) + \{a \text{ } b_i \text{ on } \textbf{every} \text{ edge } e_i \subset \Gamma(1) \times (\xi_0 = -1)\}.$$

For these  $b \in B[\text{new}]$  the dual  $\eta$ 's are: For  $b_j \in B(\text{old})$ , the same  $\eta_j$  as before, in  $X^2(\text{old})$  and, for  $b_i \subset e_i \subset \Gamma(1) \times [\xi_0 = -1]$  the dual object is  $D^2(\eta_i) = e_i \times [0 \geq \xi_0 \geq -1]$  with, of course  $\eta_i = \partial D^2(\eta_i)$ , as already said.

The BLUE collapse  $X^2[\text{new}] \searrow$  pt proceeds as follows.

- i) We delete the  $D^2(\gamma^1)[\text{NEW}]$ 's introduced above.
- ii) Then, starting from  $\Gamma(1) \times (\xi_0 = -1)$  we collapse away  $\Gamma(1) \times [0 > \xi_0 \geq -1]$  which leaves us with  $X^2(\text{old})$  which we collapse, then, normally.

**Addendum.** It also makes sense to introduce the following NEW RED 3<sup>d</sup> object

$$X^3[\text{new}] \equiv \{X^3(\text{old}) \underbrace{\cup}_{\Delta^2 \times (\xi_0 = 0)} \Delta^2 \times [0 \geq \xi_0 \geq -1] - \{\text{the interiors of the 3-cells of } \Delta^3 \subset X^3(\text{old})\}\},$$

which comes endowed with a RED 3<sup>d</sup> collapse proceeding as follows.

- i) Start with the normal collapse  $X^3(\text{old}) \searrow \Delta^2 \times (\xi_0 = 0) \subset X^3(\text{old})$ .
- ii) Then, continue with  $\Delta^2 \times [0 \geq \xi_0 \geq -1] \searrow \Delta^2 \times (\xi_0 = -1)$ .

The 2-cells which are demolished by this RED 3<sup>d</sup> collapse are exactly the

$$\{\text{old } D^2(\gamma^0) \subset X^3(\text{old})\} + \{\text{the } D^2(\Gamma) \times (\xi_0 = 0), \text{ retrograded as } D^2(\gamma^0)[\text{NEW}] \text{'s}\}.$$

End of the description of the transformation OLD  $\Rightarrow$  NEW.  $\square$

We will now move to the DOUBLED CONTEXT, where the objects introduced will be 2-dimensional only, for the time being. We continue to talk about sites  $B, R$  and of 2-cells  $D^2(\eta)$ ,  $D^2(\gamma^1)$ ,  $D^2(\gamma^0), \dots$ . There are no 3<sup>d</sup> collapses alive but, before the RED (respectively BLUE) 2<sup>d</sup> collapse can start, the  $D^2(\gamma^0)$ 's (respectively the  $D^2(\gamma^1)$ 's) have to be deleted, by *decree*.

[We do not throw away, yet, the RED 3<sup>d</sup> collapse, its ghost will be still of great use to us, although the collapse, as such will not be put into effect.]

Next comes the really important change of point of view, namely THE DOUBLING PROCESS. Still another abstract sixth axis is to be introduced, call it  $-\infty < \zeta < +\infty$  and on it we fix three points labelled  $r < \beta \ll b$  with  $|\beta - r| \ll |b - r|$ . Two copies of  $X^2 = X^2[\text{new}]$  are to be considered now, call them respectively  $X^2 \times r$  and  $X^2 \times b$ . The  $X^2$  is here our friend  $X^2(\text{new})$ . What follows next will be, for the time being, a purely abstract 2<sup>d</sup> construction inside the space

$$(13) \quad (X^2 \times r) \cup (\Gamma(\infty) \times [r, b]) \cup (X^2 \times b),$$

where  $[r, b] \subset (-\infty < \zeta < \infty)$  and where the first “ $\cup$ ” is along  $\Gamma(\infty) = \Gamma(\infty) \times r$  and the second along  $\Gamma(\infty) = \Gamma(\infty) \times b$ .

Notice that there are two kinds of edges  $e \subset \Gamma(\infty)$ , the edges  $e(B)$  which contain a  $b_i \in e$  (call them, specifically  $e(b_i)$ ) and all the others, which I will chose to call, generically,  $e(r)$ . Among the  $e(r)$ 's we will find all the edges

$$e = P \times [0 \geq \xi_0 \geq -1], \quad P \in \{\text{vertices of } \Gamma(1)\} \subset \Delta^2.$$

In the Figure 7 below, the last edges above are in the category (IV).

In the context of (13), to each edge  $e \subset \Gamma(\infty)$  corresponds a 2-cell  $e \times [r, b] \subset \Gamma(\infty) \times [r, b]$ .

We will use the notations

$$(14) \quad \begin{aligned} (D^2(c(b_i)), c(b_i)) &\equiv (e(b_i) \times [r, b], \partial(e(b_i) \times [r, b]) \\ (D^2(c(r)), c(r)) &\equiv (e(r) \times [r, b], \partial(e(r) \times [r, b])), \text{ i.e. } c = \partial D^2(c). \end{aligned}$$

Remember that, by now the old context has been replaced by the new one and, in turn, this will be replaced itself by the DOUBLED context, i.e. we perform successively the changes

$$(15) \quad \text{old} \implies \text{new} \implies \text{DOUBLE}.$$

But while the old context was only a pedagogical gimick, to be forgotten, both the NEW and the DOUBLED context (which does not quite superside the NEW one, for instance the BLUE flow on  $X^2[\text{NEW}]$  is not the restriction of the BLUE flow on  $2X^2$ ), will have to be used, both of them.

When we go to the DOUBLED context, then the higher analogue of  $X_0^2$ , the space of the 2<sup>d</sup> RED collapsing flow, will be the following 2<sup>d</sup> object where, of course  $X_0^2 \times r = X_0^2[\text{new}] \times r$ ,

$$(16) \quad 2X_0^2 \equiv (X_0^2 \times r) \cup \{\Gamma(\infty) \times [r, b] \text{ with any 2-cell } D^2(C(b_i)) \text{ deleted and replaced by a very thin tubular neighbourhood of its boundary. The newly created boundary component is re-baptized } c(b_i)\} \cup \left\{ \left( \bigcup_1^\infty D^2(\eta_\ell) \right) \times b, \text{ a space which I will call } X_b^2 \right\}.$$

Remember that at the level of our  $X_0^2 \times r \approx X_0^2[\text{new}]$  all the  $D^2(\gamma^0)[\text{NEW}] = D^2(\Gamma_i) \times (\xi_0 = 0)$  are *deleted* and actually replaced by thin tubular neighbourhoods of their boundaries. This allows us to write generically

$$(16.1) \quad \partial(2X_0^2) = \sum_k \gamma_k^0[\text{NEW}] + \sum_{b_i \in B[\text{NEW}]} c(b_i) \supset \sum_j \Gamma_j \times (\xi_0 = 0) \text{ (on the } r\text{-side)},$$

with an obvious small twist of notation. [Any 2-cell which gets deleted is replaced by a boundary collar, the exterior frontier of which occurs now in (16.1).]

The 1-skeleton of  $2X_0^2$  is the following object

$$2\Gamma(\infty) \equiv (\Gamma(\infty) \times r) \cup (\Gamma_0(\infty) \times [r, b]) \cup (\Gamma(\infty) \times b),$$

when  $\Gamma_0(\infty) \subset \Gamma(\infty)$  is the 0-skeleton of  $\Gamma(\infty)$ .

BLUE-wise, our  $2X_0^2$  is limping, reason for introducing a higher analogue of  $X^2$  too, namely the

$$(17) \quad 2X^2 \equiv (X_0^2 \times r) \cup (\Gamma(\infty) \times [r, b]) \cup X_b^2 \supset 2X_0^2.$$

The reason why, inside  $2X^2$  we use  $X_0^2 \times r \approx X_0^2[\text{new}]$ , just like for  $2X_0^2$ , and **not** the seemingly more appropriate  $X^2[\text{new}] \times r$ , will soon be crystal clear.

In the context of  $X^2[\text{new}]$ , the 2<sup>d</sup> RED and BLUE collapses were expressed in terms of sites  $R, B$  which, for  $X^2[\text{new}]$  I will denote them now by  $R_0, B_0$  and the corresponding curves  $\Gamma, C, \gamma^0$ , respectively  $\eta, \gamma^1$ . [We speak now about  $X^2[\text{new}]$  while the initial  $X^2$  call it  $X^2(\text{old})$  will never play any role any longer, and will be forgotten, whenever the contrary is not explicitly said.]

**Lemma 5.** 1) *We can endow  $2X^2 \supset 2X_0^2$  with RED and BLUE sites  $R_1 \supset R_0, B_1 \supset B_0$  and with extended system of curves so that  $2X_0^2$  should carry a 2<sup>d</sup> RED collapsing flow  $2X_0^2 \searrow \Delta^2 = \Delta^2 \times (\xi_0 = -1) \subset X_0^2 \times r$  and a 2<sup>d</sup> BLUE collapse  $2X^2 - \sum \text{int } D^2(\gamma^1)$  (to be defined in the doubled case)  $\searrow \text{pt.}$  It will turn out that the  $2X_0^2$ , as defined, is already free of  $D^2(\gamma^0)$ 's so we did not have to take, for defining its RED collapse, the same precautions as for the BLUE collapse of  $2X^2$ .*

No 3<sup>d</sup> collapse will ever be considered at the doubled level but, at the level of  $X_0^2[\text{new}]$ , but for technical reason, we will still need to invoke later on, the RED 3<sup>d</sup> collapse and a ghostly memory of it will linger at the level of our  $2X_0^2$ .

2) *Here are the explicit spots (1-handles) and curves (attaching zones of 2-handles or simply boundary components), contained in  $\Gamma(2\infty)$ , at the double level. The 1-handles (or spots) are*

$$(18) \quad R_1 \equiv \{ \text{The family } R_0 \subset X_0^2 \times r \} + \{b_i \times b\} = \{e(b_i) \times b\} + \{e(r) \times b\} = \left[ \sum_1^n R_i \times (\xi_0 = -1) \right] + \sum_1^\infty h_n,$$

with  $b_i \in B_0$  and with  $e(r)$  like in the Figure 7-(I, IV),  $e(b)$  like in Figure 7-(II, III), and, of course, with  $\{b_i \times b\} + \{e(r) \times b\} \subset X^2 \times b$ ; then

$$B_1 = \underbrace{\{b_i \times r\}}_{\text{The } B_0, \text{ living in } X_0^2 \times r} + \{b_i \times b\} + \{e(r) \times b\},$$

the last two items being common for  $R_1$  and  $B_1$ , and see here the Figures 7 and 7-bis. We have  $B_0 = B[\text{new}]$  (introduced in the context  $X^2[\text{new}]$  (12.1). End of (18).

Next we have

$$(18.1) \quad \{ \text{extended set of } C \text{'s} \} = \{C \subset X_0^2 \times r\} + \{c(r) \text{ (see (14))} + \{\eta \times b \subset X_b^2\}; \{ \text{extended set of } \gamma^0 \text{'s} \} = \{ \text{the } \gamma^0 \subset X_0^2 \times r \text{ with the } \partial\Gamma_i \times (\xi_0 = 0) \text{ included} \} + \{c(b)\} \text{ (see (14)); } \{ \text{extended set of } \eta \text{'s} \} = \{ \eta \times b \subset X_b^2 \} + \underbrace{\{c(r)\} + \{c(b)\}}_{\text{in } \Gamma(\infty) \times [r, b]}; \{ \text{extended set of } \gamma^1 \text{'s} \} = \{\Gamma_i \subset (\xi_0 = 0 \text{ and } -1)\} + \{C_j\}.$$

So, **all** 2-cells of  $X_0^2 \times r$  are, BLUE-wise speaking,  $D^2(\gamma^1)$ 's, when we go to  $2X^2$ , and this is the reason to define the  $2X^2$  in (17) as we did. The 2<sup>d</sup> BLUE flow of  $2X^2$  is now mute on the piece  $X_0^2 \times r$ .

Here are some EXPLANATIONS concerning the lines 2 (the  $\gamma^0$ 's) and 4 (the  $\gamma^1$ 's) in (18.1). None of the  $\gamma^0 \in \gamma^0$  (of  $X_0^2 \times r = X_0^2[\text{new}]$ ) are physically there, neither in  $2X_0^2$  nor in  $2X^2$ . But the ghost of the RED 3<sup>d</sup> collapse (where the  $\gamma^0$ 's do appear) will be used, reason for including those  $\gamma^0[\text{NEW}] \subset X_0^2[\text{NEW}]$  in our formulae. The  $\{D^2(\gamma^0)\} \ni D^2(c(b)) \subset 2X^2$  and they have to be deleted in  $2X_0^2$  before the RED 2-collapse can start. [The  $D^2(c(b))$ 's are  $\{\text{extended } D^2(\gamma^0)\}$ 's]. Similarly, the  $D^2(\text{extended } (\gamma^1))$  have to be deleted so as to proceed to the BLUE 2<sup>d</sup> collapse of  $2X^2 \searrow \text{pt.}$  With all these things, here are the two basic RED and BLUE geometric intersection matrices, at levels  $2X_0^2 \subset 2X^2$ . It is these matrices which define our two 2<sup>d</sup> collapsing flows

(19)  $\{\text{extended set of } C\text{'s}\} \cdot \{\text{extended set of } h\text{'s}\} = \text{easy id} + \text{nilpotent}$ , where  $\sum h \text{ (extended)} \equiv R_1 - \Gamma(1) \times (\xi_0 = -1)$ , and remember that  $\sum_1^n R_i \times (\xi_0 = -1)$  are not to be mixed up with the bigger set  $\sum h_n \subset R_1$ ; then  $\{\text{extended set of } \eta\text{'s}\} \cdot \{\text{extended set of } B_1\text{'s}\} = \text{easy id} + \text{nil}$ . End of (19).

Here are the explicit dualities established by the diagonal  $\delta_{ij}$ 's of our matrices, between sets of 1-handles and sets of attaching curves of 2-handles.

(20) (RED duality)  $\{C_i \subset X_0^2 \times r\} \longleftrightarrow \left(\sum_1^\infty h_n\right) \cap X_0^2 \times r$ , occurring in the  $R_0 - B_0$ , or  $R_0 \cap B_0$ , and see here Figure 7-(I), respectively 7-(II),

$$\{\eta_i \times b\} \longleftrightarrow \{b_i \times b\} \quad \text{and} \quad \{c(r)\} \longleftrightarrow \{e(r) \times b\}.$$

Here the  $\{b_i \times b\}$ ,  $\{e(r) \times b\}$  are the  $\left(\sum_1^\infty h_n\right) \cap X_b^2$  and they occur, respectively, the Figures (7-II, III) + (7.bis) and 7-(I, IV). All of them are  $R_1 \cap B_1$ 's.

(21) (BLUE duality)  $\{\eta_i \times b\} \longleftrightarrow \{b_i \times b\}$  (and this is the BLUE duality of  $X^2[\text{new}]$ , **transported** from  $X^2 \times r$  to  $X^2 \times b$ ), then

$$\{c(r)\} \longleftrightarrow \{e(r) \times b\}, \quad \{c(b_i)\} \longleftrightarrow \{b_i \times r \in e(b) \times r\}.$$

The first of these occurs in Figure 7-(I, IV) and the second in 7-(II, III) + 7.bis. With all these things the two 2<sup>d</sup> RED and BLUE flows, on  $2X_0^2$ , respectively on  $2X^2$ , are completely defined. But we will explicit them even more.

(21.1) The BLUE 2<sup>d</sup> flow goes like follows, for  $2X^2$

$$\underbrace{\{b_i \times r\}}_{\substack{\text{no internal blue} \\ \text{flow lines among} \\ \text{these guys}}} \longrightarrow \underbrace{\{b_i \times b\}}_{\substack{\text{All the blue flow lines of} \\ X^2[\text{new}], \text{ given by } \eta_j \cdot b_j \\ \text{concerns } \textbf{these} \text{ guys}}} \longrightarrow \underbrace{\{e(r) \times b\}}_{\substack{\text{no internal blue} \\ \text{flow lines among} \\ \text{these guys}}}.$$

To make things completely explicit here are also the off-diagonal terms in our geometric intersection matrices.

(21.2) (RED case) On  $X_0^2[\text{new}] = X_0^2 \times r$  we have the  $C \cdot h$  of  $X_0^2[\text{new}]$ , with the corresponding off-diagonal terms. On  $X_b^2$  we have the  $\eta \cdot b$  of  $X^2[\text{new}]$ , with the corresponding off-diagonal terms. For the part of the matrix concerning  $c(r)$  the situation is completely readable in Figure 7-(I, IV). In the cases of 7-(I, IV) we have off-diagonal terms  $e(r) \times b \xrightarrow{C \cdot h} e(r) \times r$  and, while in (I) things continue with  $C \cdot h \mid X_r^2$ , at  $e(r) \times r$  in (IV) the RED flow stops (Dead End).

(21.3) (BLUE case) On  $X_b^2$  we have the  $\eta \cdot b$  of  $X^2[\text{new}]$  with the same off-diagonal terms as in the RED case. For  $c(r)$ ,  $c(b)$  the situation is readable respectively as Figure 7-(I, IV) OR ((7-(II, III)) + (7.bis)). In the cases 7-(I) and 7-(IV) things are exactly as in the RED case. In the situation ((7-(II, III)) + (7.bis)), we have  $e(b) \times r \xrightarrow{\eta \cdot B} e(b) \times b$ , as off-diagonal term.

So, keep in mind that, RED-wise, at level  $2X_0^2$ , on the  $X_0^2 \times r$  side we just leave the  $C \cdot h$  of  $X_0^2[\text{new}]$ , while the  $\eta \cdot B$  of  $X^2[\text{new}]$  is transported on  $X_b^2$ , RED-wise, not only BLUE-wise.

Formulae (20), (21) and Figure 7 should help making explicit our two collapses. Here is, in detail, the RED  $2^d$ -collapse of  $2X_0^2$  (at the level of which the  $D^2(\gamma^0)$ 's are already deleted).

- ) We collapse away  $X_b^2$  using the sites  $e(b) \times b = b_i \times b$  in Figure 7-(II, III), 7.bis, which are free on their left side. This leaves us free to unleash the flow  $\eta \cdot B$  [of  $X^2[\text{NEW}]$ ] on the  $X_b^2$ , where it has been transported and demolish the  $X_b^2$ . There is, of course, no RED action on  $\Delta^2 \times (\xi_0 = -1)$  but, for the  $b \in B_0 \cap \Delta^2 \times (\xi_0 = -1)$ , the  $b \times b \in \Gamma(\infty) \times b$  partake into the RED flow of  $X_b^2$ . [See here the [BCGF] in Figure 19.1.] Also, we talk here in terms of a mythical “infinite collapse”; but what one should read, in real life is actually the following story: when we consider  $\Gamma(\infty) \times b \supset \{b_i \times b\}$ , for all the  $b_i \in B_0$ , and to which the  $D^2(\eta) \times b$  are attached, this is the same thing, up to isomorphism as  $X^2(\text{old}) - \sum D^2(\gamma^1)$  with  $B_i \subset \Gamma(\infty)$ . So all the  $\eta \cdot B$  game can be played on the  $X_b^2$  side now, and leave us, on the  $b$ -side with only  $\Gamma(\infty) \times b - \{b_i \times b\}$  alive. From here on we can move to the next ••).
- ) Next we collapse the pairs  $(e(r) \times b, e(r) \times [r, b])$ , occurring in the Figures 7-(I and IV). They are now free on their right side, because of •). By now, also, after the collapse in ••), only  $(\Gamma(\infty) \times r) \cup \Gamma_0(\infty) \times [r, b] \subset 2\Gamma(\infty)$ , is alive, as far as the 1-skeleton is concerned, plus the  $X_0^2 \times r$ , of course.
- ) Finally, we collapse normally

$$\{X_0^2 \times r = X_0^2[\text{new}]\} - \sum_1^{\bar{n}} D^2(\Gamma_i) \times (\xi_0 = 0) \searrow \Delta^2 = \Delta^2 \times (\xi_0 = -1).$$

End of the RED collapse.

Here is now, again in detail, the BLUE  $2^d$  collapse of  $2X^2 \supseteq 2X_0^2$ :

- ) We start by eliminating all the int  $D^2(\text{extended } \gamma^1)$ , which includes now all the  $D^2$ 's of  $X_0^2[\text{NEW}] \approx X_0^2 \times r$ , including the  $D^2(\Gamma_j) \times (\xi_0 = -1)$ , extending thus the (12.2). This is an important feature of the DOUBLING.
- ) Then we collapse away the pairs  $(e(b) \times r, e(b) \times [r, b])$  which are now free on their left side, Figures 7-(II, III), 7.bis.
- ) Afterwards, we collapse normally  $X_b^2$  starting at the site  $e(b) \times b$ , which are by now free. In other terms, we perform the BLUE collapse of  $X^2[\text{new}]$ , transported to the  $X_b^2$ -side.
- ) Finally, we collapse away the  $(e(r) \times b, e(r) \times [b, r])$ , in Figure 7-(I, IV), and this leaves only with a tree alive. End of the collapsing story.

3) (**Punch line**) Here are the two goals which the DOUBLING has achieved, namely

A) We have now, in agreement with (12.2)

$$(21.A) \quad \{\Gamma \times (\xi_0 = -1)\} \subset \{\text{extended } \gamma^1 \text{'s}\}.$$

[Actually already at the level  $X^2[\text{new}]$ ,  $\{\Gamma \times (\xi_0 = -1)\} \subset \{\gamma^1\}$ , and this is not violated by the passage  $\text{NEW} \Rightarrow \text{DOUBLE}$ .]



(21.B) *At the level of  $2X^2$  the RED and BLUE never cut transversally through each other. When a 2-cell  $\sigma^2 \subset 2X^2$  contains both a RED and a BLUE collapsing arrow, they coincide. When before the DOUBLING they might have happily cut through each other, the RED and BLUE arrow become now parallel, or disjointed, if you wish.*

4) (**A strategic decision**) *All the edges  $e \subset \Gamma(1) \times (\xi_0 = -1)$  contain a  $b = b(e) \in B_1$ , so they are in the same boat as the edges  $e(b)$  from the Figures 7-(II, III). Moreover, there are no other edges at  $\xi_0 = -1$ .*

*Now, our **strategic decision** is that for exactly all the edges at  $\xi_0 = -1$  we completely erase the shaded collar which is visible in the Figures 7-(II, III). What this achieves is the following (and see here Figure 7.bis, replacing 7-(II, III) at  $\xi_0 = -1$ ).*

(21.C) *The only  $2^d$  pieces not in  $\Delta^2 \times (\xi_0 = -1)$  but adjacent to it, come from  $\Gamma(1) \times [0 \geq \xi_0 \geq -1]$ . In particular, there is no  $2^d$  piece  $A^2 \subset \Gamma(\infty) \times [r, b]$  adjacent to  $\Delta^2 \times (\xi_0 = -1)$ .*

End of Lemma 5.

**Very important remark.** We may sometimes have to use subdivisions which normally would have to preserve the GPS system. But, when we apply such a subdivision, let us say to  $X^2[\text{new}]$ , then  $D^2(\Gamma), D^2(C), D^2(\eta)$  break into several 2-cells with the same labels, while for a normal subdivision we find

$$D^2(\gamma^0) \implies \{\text{a unique smaller } D^2(\gamma^0) \text{ and many small } D^2(C)\text{'s}\},$$

$$D^2(\gamma^1) \implies \{\text{a unique smaller } D^2(\gamma^1) \text{ and many small } D^2(\eta)\text{'s}\}.$$

The last line above comes with the big potential danger of destroying the feature (21.A) which we will very much need. But we will **need** to subdivide  $\Delta^2 \times (\xi_0 = -1)$  too, in section V below (CONFINEMENT). And then, so as to preserve (21.A) we will similarly subdivide  $\Delta^2 \times (\xi_0 = 0)$  too, and for **any** vertex  $P \in \Delta^2$  we will add a line  $P \times [0 \geq \xi_0 \geq -1]$ , see here how we proceed in Figure 30 below. The whole of  $\Delta^2 \times [0 \geq \xi_0 \geq -1]$  is subdivided uniformly there. The same important remark is valid after DOUBLING, of course.  $\square$

$\Delta^2 = \Delta^2 \times \{\xi_3 = 1\}$ , living inside  $X^{\text{old}}$ . The point here is that  $\Delta^2_{\text{new line}}$  is codimension zero in  $X^4 = R^4$ . This is in contrast with the situation when the technology of the present paper is applied to the homotopy 3-ball  $\Delta^3$  for showing that  $\Delta^3 \times I \in \text{GSC}$ . One starts then from

$$\Delta^3 \times I \subset \text{Im}\{\Delta^3 \times I \xrightarrow{\neq} \infty \xrightarrow{\neq} (S^2 \times D^2)\} \in \text{GSC} \quad (\text{see [Po2]}).$$

with a  $\Delta^3$  which is then *codimension one*.

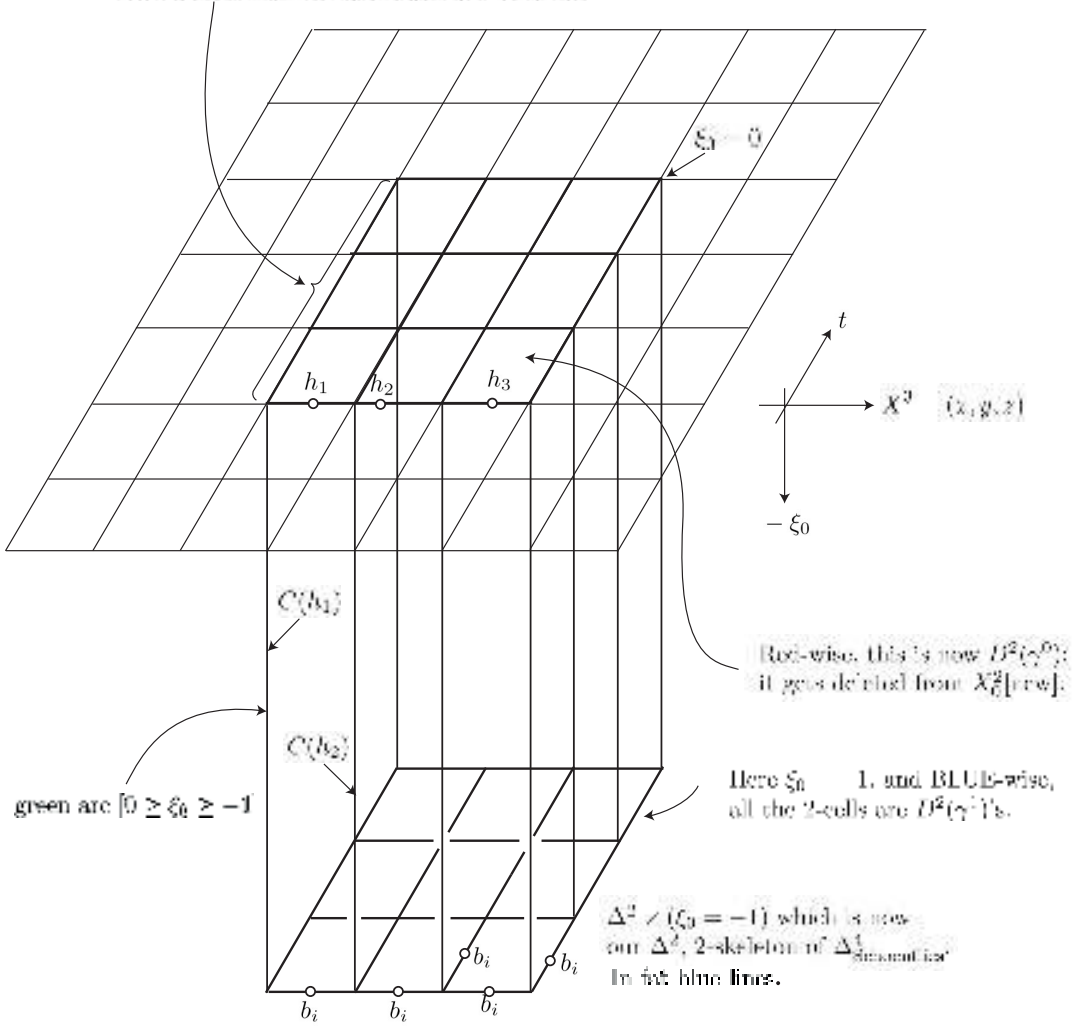


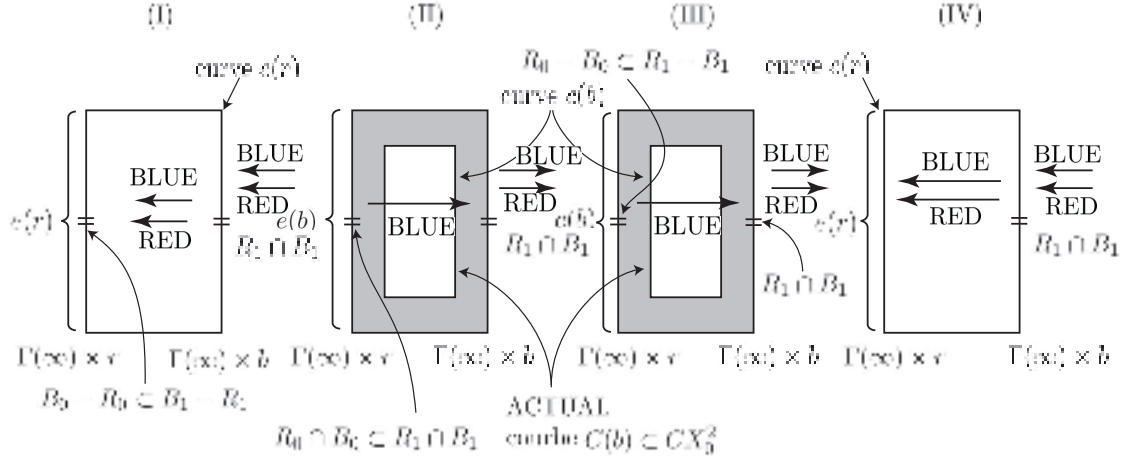
Figure 6.

Schematical representation of  $X^2[\text{new}] \supset X_0^2[\text{new}]$ . Our drawing is, oversimplified, at least in two respects: we have failed to represent the GPS cellular structure and, of course also, the dimensions are highly unrealistic. But it should be stressed that *each* vertex of  $\Delta^2 \times (\xi_0 = -1)$  is joined via a green arc to correspondant vertex in  $\Delta^2 \times (\xi_0 = -1)$ . In this figure we see the  $\Gamma(1) \times (\xi_0 = 0)$  represented in fat black lines (= —), the  $[0 \geq \xi_0 \geq -1] \times P$ 's in green lines (= —) and the  $\Gamma(1) \times (\xi_0 = -1)$  in fat blue lines (= —). Every edge  $e_i \times (\xi_0 = -1)$  carries a  $b_i \in B_0$ , every  $e_j \times (\xi_0 = 0)$  carries a  $h_j$  with the dual cell being  $D^2(C_j) = e_j \times [0 \geq \xi_0 \geq -1]$

and the edges  $P \times [0 \geq \xi_0 \geq -1]$  are free of  $R \cup B$ . The edges  $e \times (\xi_0 = 0)$  may carry  $b \in B_0$ 's too.

Figure 7 illustrates what goes on along  $\Gamma(\infty) \times [r, b] \subset 2X^2 \supset 2X_0^2$ . We see a 2-cell  $e \times [r, b]$ , when  $e \subset \Gamma(\infty)$  is an edge, with  $e \subset X^2[\text{new}]$  and there are four cases to consider, namely the following

(Case I)  $R_0 - B_0 \in e$ , (case II)  $R_0 \cap B_0 \in e$ , (case III)  $B_0 - R_0 \in e$ , (case IV)  $R_0 \cup B_0 \notin e$ .



**Figure 7.**

The discs  $e \times [r, b] \subset 2X^2$ , when far from  $\xi_0 = -1$ , where Figure 7.bis replaces the present Figures 7-(II, III). At the level  $2X_0^2$ , only the shaded part of II, III survives.

LEGEND:  $\#$  = spot, in  $R_1 \cup B_1$ ,  $\xleftarrow{\text{BLUE}}$  = BLUE 2<sup>d</sup> collapsing flow (for  $2X^2$ ),  $\xleftarrow{\text{RED}}$  = RED 2<sup>d</sup> collapsing flow (for  $2X_0^2$ ). These flows are represented here only to the extent they affect the  $\Gamma(\infty) \times [r, b] \subset 2X^2$ . More RED arrows may actually also leave from  $e(r)$  in (I), (IV) to the left, in  $X_0^2 \times r$ , but we have failed to represent this here. The  $X_b^2$  lives to the right of these figures. In (I), (IV), the outer curve of the rectangle is of the type  $c(r)$  while in (II), (III) it is of type  $c(b)$ ; but then when going from  $2X^2$  to the smaller  $2X_0^2$ , the name is inherited by the corresponding smaller curve in  $\partial(2X_0^2)$ . The RED arrows inside (I), (IV) correspond, in terms of the dualities from (20), to  $\{c(r)\} \xleftrightarrow{\approx} \{e(r) \times b\}$ . The BLUE arrows inside (I), (IV) corresponds to the same  $\{c(r)\} \xleftrightarrow{\approx} \{e(r) \times b\}$  (which occurs in both geometric intersection matrices). The outgoing BLUE and RED arrows of (II), (III) correspond to the  $\{\eta_i \times b\} \xleftrightarrow{\approx} \{b_i \times b\}$  in our matrices. In the context of (I), (IV) the BLUE and RED incoming arrows in  $e(r) \times b \in R_1 \cap B_1$ , just means that the corresponding edges  $e(r) \times b$  may receive such arrows from  $X_b^2$ , without sending themselves any, back into  $X_b^2$ . To the left of the  $e \times r$  in (I) or (II) we just have the normal RED flow inside  $X^2[\text{new}]$ , which we did not represent here by arrows, as already said.

Any edges  $e \subset \Delta^{(2)} \times (\xi_0 = -1)$  would find itself normally as an  $e(b)$  in (II), (III). But since now the whole collar is deleted, in agreement with 4) in Lemma 5, we have redrawn the situation at  $(\xi_0 = -1)$  in Figure 7.bis. Finally, for any vertex  $P \in \Gamma(1)$ , the  $e = P \times [0 \geq \xi_0 \geq -1]$  is among the  $e(r)$ 's in (IV).

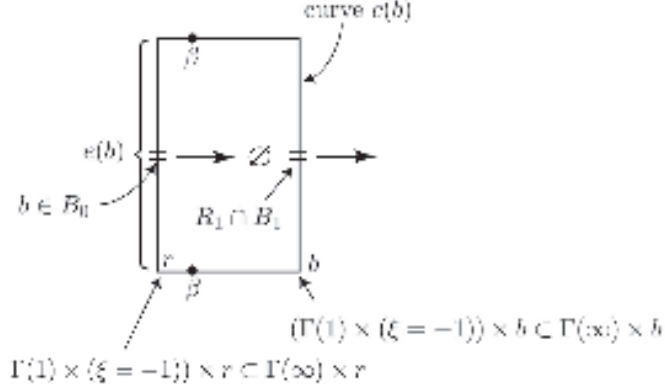


Figure 7.bis

Here is what the Figures 7-(II, III) become when the corresponding  $c(b) \subset \Delta^2 \times (\xi_0 = -1)$ . **All** the edges of  $\Delta^2 \times (\xi_0 = -1)$  are like this  $e(b)$ .

LEGEND: — = the curve  $c(b)$  for  $b \in \Delta^2 \times (\xi_0 = -1)$ .  $\longrightarrow$  = BLUE flow at doubled level. No collar is surviving here, unlike in Figures 7-(II, III). All arrows are here BLUE.

### 3 Four dimensional thickenings and compactifications

We start now from

$$2X_0^2 \supset X_0^2 \times r \approx X_0^2[\text{NEW}] \supset \Delta^2 \times (\xi_0 = -1) = \{\Delta^2, \text{2-skeleton of } \Delta_{\text{Schoenflies}}^4\},$$

and we want to define  $4^d$  regular neighbourhoods for  $2X_0^2$ . Now, there is no a priori given DIFF 4-manifold into which  $2X_0^2$  is embedded, or even immersed, and we will start by defining, **by decree**, local  $4^d$  thickenings. Then, going from local to global, we glue together these pieces, making use of appropriate framings, when 2-handles are to be added. This construction will be restricted by two conditions which it will **have to satisfy**

(22.A) (Compatibility with Schoenflies) So as not to loose the connection with the Schoenflies issue, we will insist that  $\{N^4(2X_0^2) \text{ defined by glueing the chosen local piece above}\} \mid \Delta_{\text{DIFF}}^2 = \{\text{the normal } N^4(\Delta^2) \text{ from Theorem 2, i.e. the normal } 4^d \text{ regular neighbourhood of the } \{2^d \text{ skeleton } \Delta^2 \text{ of } \Delta_{\text{Schoenflies}}^4\} \subset \Delta_{\text{Schoenflies}}^4\}$ .

(22.B) Compatibility with the  $2^d$  RED collapsing: We do insist that when the  $N^4(2X_0^2)$  is expressed as

$$\underbrace{N^4(2X_0^2) = N^4(2\Gamma(\infty))}_{N^4 \text{ (infinite tree) + } \left\{ \text{The 1-handles } \sum_1^n R_j + \sum_1^\infty h_i \right\} \text{ (see 19)}} + \sum \{2\text{-handles } D^2(c) \text{ corresponding to the } \{ \text{extended set of } D^2(c) \text{'s} \} \text{ (18.1)}\} + \sum_1^{\bar{n}} D^2(\Gamma_i),$$

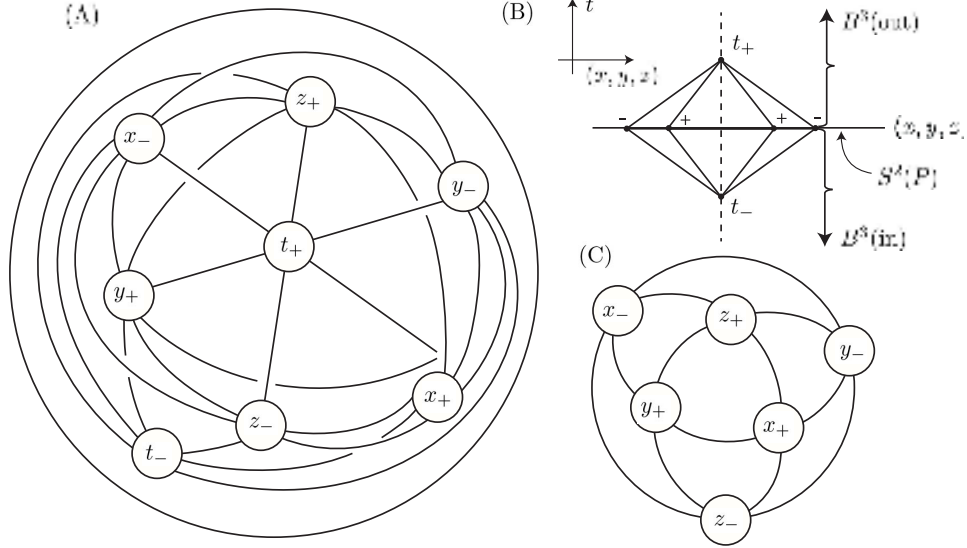
then we should find exactly the same geometric intersection matrix

$$C \cdot h = \text{id} + \text{nilpotent (of the easy type)},$$

as in the first line of (19).

**Remark.** There is, a priori, another possibility to proceed, in order to get our  $N^4(2X_0^2)$ , namely to start by choosing some appropriate immersion of  $2X_0^2$  into some ambient 4-manifold ( $\Delta_1^4$  in (2), for instance) and then to take its  $4^d$  regular neighbourhood.

But I found the present manner of proceeding much more convenient, in particular more efficient when discontinuous **changes on local topology** always compatible with (22.A) + (22.B), hence NOT contradicting the **global** topology, will be necessary. End of Remark.



Together with the triangle  $d(x_-, y_-, z_-)$ , this defines a 2 square  $S^2(P) \subset \partial N^3(P) = S^3(P)$ ; see the (B) and.

**Figure 8.**

We see here, in (A), the trace  $(X_{\text{cubical}}^2 \mid P) \cap S^3(P)$ .

EXPLANATIONS: The (A) is gotten by suspending the  $S^2(P)$  from (C), from the  $\bigoplus_{+t}$  and  $\bigoplus_{-t}$ , like it is suggested in (B). Actually, what (B) suggests is a SPLITTING

$$S^3(P) = \partial N^4(P) = B^3(\text{out}) \bigcup_{S^2(P)} B^3(\text{in}), \text{ with } \bigoplus_{-t} \subset B^3(\text{in}), \bigoplus_{+t} \subset B^3(\text{out}),$$

while in (C) we have redrawn a detail of the main (A), which corresponds exactly to

$$S^2(P) \equiv \bigcup \{\text{the eight triangles } d^2(\text{space, space, space})\}.$$

But there is also a second SPLITTING, by  $S_{\infty}^2(P) \equiv S^3(P) \cap \Sigma_{\infty}^2$ , with  $\Sigma_{\infty}^2$  like in (27), to be discussed later,

$$S^3(P) = B^3(-) \bigcup_{\Sigma_{\infty}^2} B^3(+);$$

what we see in (A) is actually  $B^3(\text{in}) \cup B^3(-)$ , with the interaction between these two splittings of  $S^3(P)$ , by  $S^2(P)$  and by  $\Sigma_{\infty}^2$  (or rather its trace  $S_{\infty}^2$  on  $S^3(P)$ ), is suggested in the Figure 9.bis.

As a preliminary for the  $N^4(2X_0^2)$  which will have to be constructed from scratch for our  $2X_0^2$ , hence not coming with any a priori God-given embedding into some given 4-manifold, I will consider now

$$X_{\text{cubical}}^2 \equiv \{\text{the 2-skeleton of the standard cubical cell-decomposition of } R^4 = R^3 \times R,$$

$$\text{with } R^4 = (x, y, z, t)\},$$

which certainly comes with the obvious embedding  $X_{\text{cubical}}^2 \subset R^3 \times R$ , from which the  $N^4(X_{\text{cubical}}^2)$  is defined. If  $P \in X_{\text{cubical}}^2$  is a vertex, we have

$$N^4(P) \equiv N^4(X_{\text{cubical}}^2 \mid P) = \{\text{the } N^4(\text{germ } X_{\text{cubical}}^2 \mid P)\}.$$

This comes with  $N^4(P) = B^4$ ,  $S^3(P) \equiv \partial N^4(P)$ .

The  $N^3((X_{\text{cubical}}^2 \mid P) \cap S^3(P)) \subset S^3(P)$  is a collection of small 3-balls  $b^3(x_{\pm}), \dots, b^3(t_{\pm})$ , occuring as  $(x_{\pm}), \dots, (t_{\pm})$  in Figure 8. Here

$$S^2(P) \equiv S^3(P) \cap \{\text{the } (x, y, z)\text{-coordinate hyperplane}\}$$

induces the splitting  $S^3(P) = B^3(\text{out}) \cup_{S^2(P)} B^3(\text{in})$ , with  $b^3(t_+) \subset B^3(\text{out})$ ,  $b^3(t_-) \subset B^3(\text{in})$ . Figure 8-(A)

is a rigorously **correct** representation of  $(S^3(P), (X_{\text{cubical}}^2 \mid P) \cap S^3(P))$ ; one can think of the 3-ball inside which the drawing lives as being  $B^3(\text{in}) \cup B^3(-)$ , see here Figure 9.bis too.

Of course, also, nice small isotopies are allowed. For instance, the  $b^3(t_-)$  which, for reasons of graphical commodity has been pulled a bit to the side, could come just under  $b^3(t_+)$  so that these should be a line:  $\{\text{observer's eye}\} \text{ --- } b^3(t_+) \text{ --- } b^3(t_-)$  (see (B) too). It is this move which is used when in Figure 9 we change  $(B_+)$  into  $(B_-)$ . The  $(X_{\text{cubical}}^2 \mid P)$  occuring above, is the germ at the origin  $0 \in R^4$  of the six coordinate planes. Each triangle  $d^2$  (space-time, space-time, space-time) stands for a corner of  $X_{\text{cubical}}^2 \mid P$ , and the  $d^2$ 's are disjoint except for their common edges or vertices. In our Figure 8-(A), the **arcs** joining two  $b^3$ 's are pieces of the curves along which 2-handles get attached.

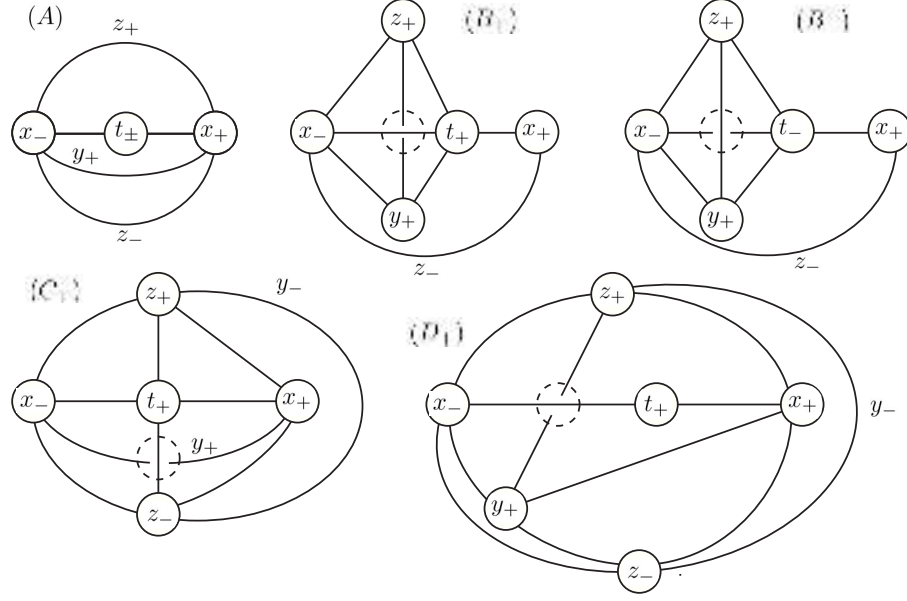
**The reconstruction of  $N^4(X_{\text{cubical}}^2)$** , out of the correct Figures 8-(A), for the various vertices  $P$ . This is done in two steps.

I) If  $P_1, P_2$  are two adjacent vertices, then in the 1-skeleton  $X_{\text{cubical}}^2$  some  $b^3(u_{\pm}) \subset S^3(P_1)$  communicates with  $b^3(u_{\mp}) \subset S^3(P_2)$ , and we will join the two via a 1-handle. This way we have reconstructed the  $N^4(X_{\text{cubical}}^1)$ , which should be, of course orientable.

II) From the arcs  $\subset \bigcup_P S^3(P)$ , joined along the lateral surfaces of the 1-handles above, in a canonical manner, we get a

$$\{\text{link}\} \subset \partial N^4(X_{\text{cubical}}^1)$$

to which, with appropriate framings we add 2-handles. This **is** the reconstruction of  $N^4(X_{\text{cubical}}^2) \subset R^3 \times R = R^4$ . But the  $X_{\text{cubical}}^2$  is only an intermediary tool, since what really interests us is the real-life GPS structure, and that will be used for  $2X_0^2$ . But then, the GPS structure is compatible with the cubical one: One moves from one to the other (cubical  $\Rightarrow$  GPS) via some deletions and simple isotopies. And so, as long as  $N^4(\Delta^2) = N^4(2X_0^2) \mid \Delta^2$  stays compatible with  $N^4(X_{\text{cubical}}^2)$ , we are sure that the (22.A) is satisfied. Otherwise, there are no restriction, outside of  $\Delta^2 = \Delta^2 \times (\xi_0 = -1)$ , as far as the framings are concerned.



**Figure 9.**

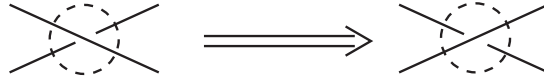
The local models for the real-life GPS  $N^4(P) \equiv N^4(2X_0^2(\text{GPS})) \mid P$ , with a  $N^4(2X_0^2(\text{GPS}))$  which is still to be defined but for which the present figures settles the local models. By appropriate permutation of space-time letters, all the local GPS models can be deduced from the ones drawn here. The present figure should be compared to Figure 8-(A), but for typographical commodity we have omitted the outer surrounding circle. The present drawings only refer to the  $(x, y, z, t)$  part of the  $N^4(2X_0^2(\text{GPS}))$ , and they are certainly compatible with Figure 8-(A). This figure will have to be embellished with the contributions coming from the axes  $\xi_0$  and  $\zeta$ . The typical embellished figure is 24.

The present (A), (B $_{\pm}$ ), (C $_{+}$ ), (D) correspond, respectively, to the (A), (B), (C), (D) in Figure 4, with the  $t_{\pm}$  added. The lines  $[z_{+}, y_{+}]$ ,  $[y_{+}, z_{-}]$ ,  $[z_{-}, z_{+}]$  are ORANGE, the others are BLACK.

**Additional explanations for the Figure 9.** Each of the five little drawings in the figure which is displayed here, once filled in with the appropriate 2-cells, with disjointed interiors, bounded by the closed curves in the figure, can be recognized as being a detail which appears in the Figure 8-(A) too. So, Figure 9 is compatible with 8-(A).

All the configurations drawn here embed, isometrically in Figure 8-(A) respecting the presently drawn  $b^3$  (space-time)'s. When the corresponding line in Figure 8-(A) has a  $b^3$  (space-time) not part of the GPS local model, then one wrote the corresponding letter next to the corresponding line.

We should also have Figures 9-(C $_{-}$ ), (D $_{-}$ ) gotten from the drawn (C $_{+}$ ), (D $_{+}$ ), by letting the lines with a dotted loop over them cross, in the following pattern



Figures 9 are used for reconstructing the GPS  $N^4(X_0^2 \times r)$ , or at least the  $N^4(\Delta^2 \times (\xi_0 = -1))$ , on the same lines as the reconstruction of the  $N^4(X_{\text{cubical}}^2)$ . And, as already said, a long as our local models are

compatible with Figure 8-(A), **and** we respect the framings of  $N^4(\Delta^2)$ , (see the context of the law II for reconstructing  $N^4(X_{\text{cubical}}^2)$ ), our condition (22.A) is automatically satisfied.

In the context of Figure 9 there is also a  $(C_-)$ , which we have not drawn, and this is gotten from  $(C_+)$  by changing  $t_+ \Rightarrow t_-$  and then using the obvious crossing operation at the spot surrounded by the dotted circle. Moving now to  $(D_+)$ , which is related to Figure 4-(D), I stress that we are here in the context of formula (6), with the time directions past/future obeying the scheme presented in Figure 9.1. In this context,  $(D_+)$  corresponds to  $t = t_{2i-1}$  and if we move to  $(D_+^*)$ , we get to  $t = t_{2i}$ , with a change BLACK  $\Leftrightarrow$  ORANGE.

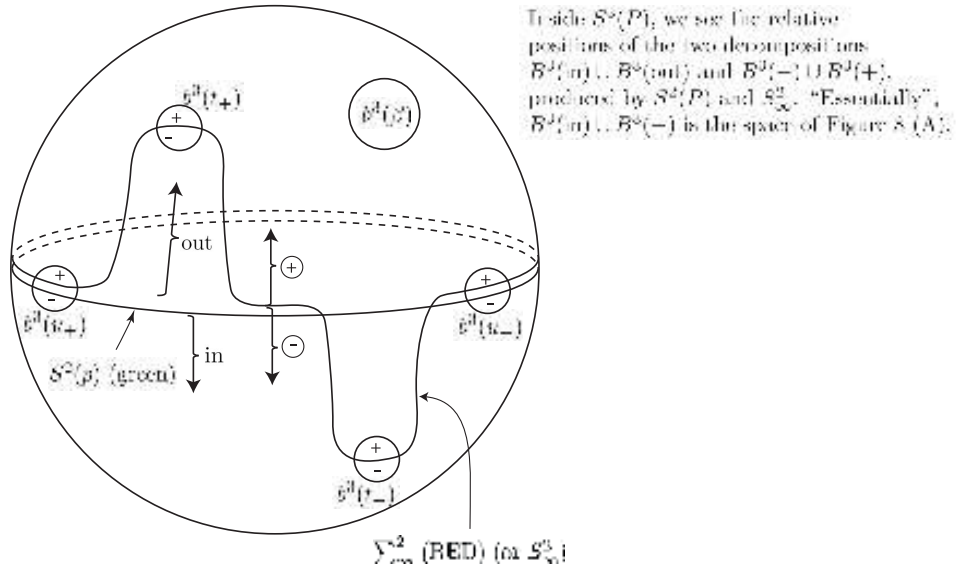


Figure 9.bis.

With one dimensionless we suggest here the two splittings  $S^3(P) = B^3(\text{in}) \cup_{S^2(p)} B^3(\text{up})$ ,  $S^3(P) = B^3(-) \cup_{\sum_2} B^3(+)$ , together.

Then there are also  $(D_{\pm}^{(*)})$  figures, gotten like  $(C_+) \Rightarrow (C_-)$ . The present 9- $(D_+)$  corresponds to 4-(D), with the same colours and with a  $t_+$  added. So we go now from  $3^d$  to  $4^d$ . With this, also, the "u" in Figure 4-(D) becomes now a vertex, bringing an ORANGE CONTRIBUTION to the 1-skeleton. End of explanations.

We move now to the still to be defined  $N^4(2X_0^2)_{\text{GPS}}$ , actually an object to be defined ex-nihilo, with the only restriction that the conditions (22.A), (22.B) should be satisfied. We start by looking at

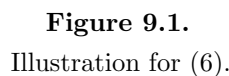
$$X_0^2[\text{new}] = X_0^2 \times r \supset X_0^2[\text{new}](\text{truncated}) \equiv X_0^2[\text{new}] - ((\Gamma(1) \times [0 > \xi_0 \geq -1]) \cup \Delta^2 \times (\xi_0 = -1)).$$

As long as we forget the coordinate  $t$ , the local models for  $X_0^2[\text{new}](\text{truncated})$  are the (A), (B), (C), (D) from Figure 4. Realistically, a  $t_+$  OR a  $t_-$  will have to be added, never both, according to the pattern from Figure 9.1. With this addition, the Figures 4 turn into 9.

The  $X_{\text{cubical}}^2 \mid P$  is the standard form

$$\{[x, y] \cup [y, z] \cup [x, z] \cup [x, t] \cup [y, t] \cup [z, t]\} \mid P.$$





33

We introduce now

$$(26.1) \quad \Gamma_1(\infty) \equiv \Gamma(\infty) \times r \cup \Gamma_0(\infty) \times [r, \beta]$$

where  $\Gamma_0(\infty) \subset \Gamma_1(\infty)$  is the set of vertices), and  $N^4(\Gamma_1(\infty)) \equiv \{\text{the (abstract) } 4^{\text{d}} \text{ regular neighbourhood of } \Gamma_1(\infty) \cong \Gamma_1(\infty) \times r \subset 2X_0^2\}$ . I will give now a useful

**Model for  $N^4(2\Gamma(\infty))$ .** We will concentrate now on  $\Gamma_1(\infty)$  (27), the extension to the full  $2\Gamma(\infty)$ , afterwards, should be transparent. Our aim is to present now, in a manner which is consistent with everything said so far, the  $N^4(\Gamma_1(\infty))$  as a product

$$(27) \quad N^4(\Gamma_1(\infty)) = N^3(\Gamma_1(\infty)) \times [0, 1],$$

and we shall think of the factor  $[0, 1]$  above as being  $[r, \beta] \subset [r, b]$ . Here is a very easy general fact. Consider any graph  $\Gamma$  (like  $\Gamma = \Gamma_1(\infty)$ ). Then we have the following

$$(27.0) \quad \text{For every } N^n(\Gamma), n \geq 4, \text{ there is a regular neighbourhood } N^3(\Gamma), \text{ such that } N^n(\Gamma) = N^3(\Gamma) \times B^{n-3}.$$

Hence, there is no problem with the bare (27), but we have to chose the correct  $N^3(\Gamma_1(\infty))$  and check the compatibilities, in particular with the basic (22.A).

**Construction of the correct  $N^3(\Gamma_1(\infty))$ .** With the not yet defined  $N^4(\Gamma_1(\infty))$  the *embellished Figure 9* (with the contributions of  $\xi_0, \zeta$  added), i.e. Figure 24, we have local models of  $(X^2[\text{NEW}] \pitchfork \partial N^4(P))$  (embellished), for the vertices  $P \in \Gamma(\infty) \subset \Gamma_1(\infty)$ . The embellishment means that we are actually talking about  $(2X_0^2 \mid P) \pitchfork \partial N^4(P)$ , for  $P \in \Gamma(\infty) \times r \subset 2\Gamma(\infty)$ . We can think of the plane of the figures of type 9 as being a  $S_0^2(P)$  on which all the  $b^3(u_{\pm})$  others than  $b^3(t_{\pm})$  ride already, as in Figure 9.bis and on which all the rest of Figure 9 is projected like in a link diagram, as an immersion

$$(27.1) \quad (X^2[\text{NEW}] \pitchfork N^4(P))_{\text{embellished}} \xrightarrow{\psi} S_0^2(P).$$

This comes with sites  $u \in \{\pm x, \pm y, \pm z, \pm t, \pm \xi_0, \beta \in [r, b]\}$  and little discs  $b^2(u)$ . Our *link diagram* contains curves joining the  $b^2(u)$ 's, and at the double points of  $\psi$  (= the CROSSINGS), UP/DOWN's are specified.

We think of  $S_0^2(P)$  which, until further notice, is simply the plane of the figures of type 9, as being the  $S_0^2(P) = \partial B_0^3(P)$ , with the  $B_0^3(P)$  living on the other side of the Figures 9, 24, with respect to the observer.

In order to construct our  $N^3(\Gamma_1(\infty))$ , we start with

$$\underbrace{\sum_{P \in \{\text{vertices of } \Gamma(\infty)\}} B_0^3(P)}, \text{ coming with } b^2(u)\text{'s} \subset S_0^2(P),$$

and notice that, if for two adjacent vertices  $P_1, P_2$  we have  $b^2(u_{\pm}) \subset S_0^2(P_1)$ , then we certainly find  $b^2(u_{\mp}) \subset S_0^2(P_2)$ . If we join each  $B_0^3(P_1)$  to  $B_0^3(P_2)$ , in an orientable manner, along a  $b^2 \times [0, 1]$  with  $b^2 \times \{0\} = b^2(u_{\pm}) \subset S_0^2(P_1)$ ,  $b^2 \times \{1\} = b^2(u_{\mp}) \subset S_0^2(P_2)$ , then we get a model for  $N^3(\Gamma(\infty))$ , endowed with a  $b^2(\beta) \subset \partial N^3(\Gamma(\infty))$  for every  $B_0^3(\beta) \subset N^3(\Gamma(\infty))$ . This structure, where we will think of the  $b^2(\beta)$ 's as sticking out, a bit, like in the Figure 11 below, is our  $N^3(\Gamma_1(\infty))$ , the construction of which is by now finished, and, by construction, it is compatible with the figures of type 9.

We come now with our additional factor  $[r, \beta] \subset [r, b]$  and it is the  $[r, \beta]$  which will be our factor  $[0, 1]$  in the context of (27).

As already said, when it comes to the compatibility with (22.A), the bare (27), which is by now in place, is irrelevant. What counts is the link diagrams (27.1) for various  $P \in \Gamma(\infty) \times r$ , and their lifts to  $4^{\text{d}}$ .

It should be understood here that  $\beta \in (r, b)$  is very close to  $r$ , so that all the thin band  $\Gamma(\infty) \times [r, \beta] \subset 2X_0^2$ , is not affected by the deletion of the non-shaded areas in Figures 7-(II, III). A more detailed version of Figure 7 is presented in Figure 45, where the splitting line occurs as a dotted line. Figure 7.bis is not concerned here; it is split by the two sites marked  $\beta$  into  $\pm$  halves,  $-$  on the  $r$ -side and  $+$  on the  $b$ -side. Let us also notice here the obvious embedding

$$(27.2) \quad \Gamma_1(\infty) \subset \Gamma(\infty) \times [r, \beta]$$

and one should notice that the pairs (germ of  $2X_0^2$  at  $\Gamma(\infty), \Gamma(\infty)$ ) and (germ of  $\Gamma(\infty) \times [r, \beta]$  at  $\Gamma(\infty) \times \frac{\beta}{2}$ ,  $\Gamma(\infty) \times \frac{\beta}{2} \approx \Gamma(\infty)$ ) are completely different from each other. In terms of Figure 24, the first germ contains the complete information, while the second one only knows about those things which are adjacent to  $\beta$ . Also, while the germ of  $2X_0^2$  lives in  $N^4(\Gamma(\infty))$  and cannot be pushed into  $\partial N^4(\Gamma(\infty))$ , the germ of  $\Gamma(\infty) \times [r, \beta]$  mentioned above, can. This last fact will be used in this paper.

We finally can introduce the

$$(27.3) \quad N^4(\Gamma_1(\infty)) = \{N^3(\Gamma(\infty)) \times [r, \beta], \text{ with a } b^2(\beta) \times [r, \beta] \text{ added to } \partial N^4(\Gamma(\infty)) \text{ at each } P\}, \text{ see Figure 11.}$$

We have here the commutative diagram

$$(27.4) \quad \begin{array}{ccc} N^4(\Gamma_1(\infty)) & \stackrel{\text{DIFF}}{=} & N^3(\Gamma(\infty)) \times [r, \beta] \\ \downarrow & & \downarrow p \\ \Gamma_1(\infty) & \subset & \Gamma(\infty) \times [r, \beta]. \end{array}$$

At the level of  $N^4(\Gamma_1(\infty))$ , the  $b^2(u)$ 's of  $\partial B_0^3(P)$  become  $B^3(u) \subset \partial N^4(P)$ . Concerning the  $S_0^2(P)$  from (27.1), when we go to  $4^d$ , it gives rise to  $S_0^2(P) \times \frac{\beta}{2} = \{\text{the } S_\infty^2(P) \text{ below}\}$ .

The compatibility with (22.A) is taken care of by the explicit liftings of the link diagrams, and what we can perceive on our model (27.3) is a PROPER and proper embedding  $N^3(\Gamma(\infty)) \times \frac{\beta}{2} \subset N^4(\Gamma_1(\infty))$ . This **defines** a SPLITTING SURFACE

$$(27.5) \quad \Sigma_\infty^2 \equiv \partial(N^3(\Gamma_1(\infty)) \times \frac{\beta}{2}) \subset \partial N^4(\Gamma_1(\infty)),$$

coming with the SPLITTING

$$(27.6) \quad \partial N^4(\Gamma_1(\infty)) = \partial N_-^4(\Gamma_1(\infty)) \cup_{\Sigma_\infty^2} \partial N_+^4(\Gamma_1(\infty)).$$

Notice the connection with (27.4), namely the equation

$$p^{-1} \left( \Gamma(\infty) \times \frac{\beta}{2} \right) \cap \partial N^4(\Gamma_1(\infty)) = \Sigma_\infty^2.$$

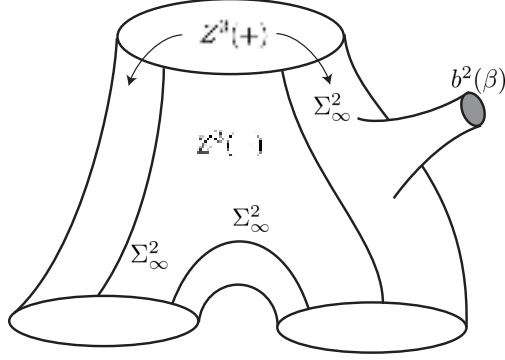
We will also use here the notations

$$Z^3(-) \equiv \partial N_-^4(\Gamma_1(\infty)), \quad Z^3(+) \equiv \partial N_+^4(\Gamma_1(\infty)).$$

The  $\pm$  convention will be fixed by setting  $\Sigma b^3(\beta) \subset \partial N_+^4(\Gamma_1(\infty))$ , with the rest of the  $b^3(\pm u)$ 's (see Figure 24) being each split by  $\Sigma_\infty^2$  into two halves, like Figure 9.bis may suggest. All the corresponding interconnections, inside the  $\partial N^4(P)$ 's, inbetween the  $b^3(\pm u)$ 's live, for the time being at least, inside the  $\partial N_-^4(\Gamma_1(\infty))$ . Later, some of them will have to be pushed into  $\partial N_+^4(\Gamma_1(\infty))$ , in a very controlled manner. Our SPLITTING

$$\partial N^4(\Gamma_1(\infty)) = Z^3(-) \cup_{\Sigma_\infty^2} Z^3(+)$$

is suggested very schematically in the Figure 11.



**Figure 11.**

The splitting by  $\Sigma_\infty^2$ , see (27.6) is suggested here, with one dimension less. Via the sites  $b^3(\beta)$  our  $N^4(\Gamma_1(\infty))$  communicates with the rest of  $N^4(2\Gamma(\infty))$ .

For each vertex  $P \in X_0^2 \times r$ , the  $\Sigma_\infty^2$  induces a splitting (see Figure 9.bis)

$$(28) \quad S^3(P) = B^3(-) \cup_{S_\infty^2} B^3(+),$$

where we use the notation  $S_\infty^2 \cong \Sigma_\infty^2 \cap \partial N^4(P)$ . This splits each  $b^3(\pm r)$  in two halves. We mean here all the  $b^3(\pm u)$ 's from the embellished Figure 24. Next, if  $P_1, P_2$  are two adjacent vertices of  $\Gamma_1(\infty)$  and  $b^3(\pm u) \subset \partial N^4(P_1)$ , then  $b^2(\mp u) \subset \partial N^4(P_2)$  and  $b^2(u) \times [-1, +1]$  joins the two  $N^4(P_i)$ 's. The splitting  $\Sigma_\infty^2$  continues along the  $\partial b^3(u) \times [-1, +1]$ . This is what Figure 11 is supposed to suggest. The splitting (28) and the splitting from the Figure 8, the  $\partial N^3(P) = B^3(\text{in}) \cup_{S^2(P)} B^3(\text{out})$  are articulated with each other like in the Figure 9.bis, which shows exactly how the splittings of  $S^3(P) = \partial N^4(P)$  by  $S^2(P)$  and by  $S_\infty^2$  (= trace of  $\Sigma_\infty^2$  on  $\partial N^4(P)$ ) interact with each other. In view of Figure 9.bis, we may also happily assume that the  $S_0^2(P) = \partial B_0^3(P)$ , which occurs in (27.1), is the  $S_\infty^2$ .

When we move from  $\Gamma_1(\infty)$  to the larger  $2\Gamma(\infty)$ , we define the following big SPLITTING which naturally extends (27.5)

$$(28.1) \quad \partial N^4(2\Gamma(\infty)) = \partial N_-^4(\Gamma_1(\infty)) \cup_{\Sigma_\infty^2} \partial N_+^4(2\Gamma(\infty)),$$

where  $\partial N_+^4(2\Gamma(\infty)) \equiv \partial N^4(2\Gamma(\infty)) - \partial N_-^4(\Gamma_1(\infty))$ . You may happily use the terminology  $\partial N_+^4(2\Gamma(\infty)) \equiv \partial N_+^4(2\Gamma_1(\infty))$ .

We will denote by  $\{\text{link}\}[\text{NEW}]$  the obvious extension of the  $\{\text{link}\}$  from (9) to  $X^2[\text{NEW}]$  with the  $D^2(\Gamma_i) \times (\xi_0 = 0)$  becoming now, RED-wise,  $D^2(\gamma_k^0)$ 's and continuing to stay, BLUE-wise, according to the case  $D^2(\eta)$  or  $D^2(\gamma^1)$ . Also, of course, the contribution of  $(\Gamma(1) \times [0 \geq \xi_0 \geq -1]) \cup (\Delta^2 \times (\xi_0 = -1))$  is to be included now too. And, at this point, remember that the  $\Gamma(1) \times [0 \geq \xi_0 \geq -1]$  is made out of  $D^2(C)$ 's RED-wise and of  $D^2(\eta)$ 's BLUE-wise, while the  $D^2(\Gamma_i) \times (\xi_0 = -1)$  (our  $D^2(\Gamma_i)$ 's, now) are  $D^2(\gamma^1)$ 's BLUE-wise. With this, we get, for the time being, the inclusion

$$(28.2) \quad \{\text{link}\}[\text{new}] \subset \partial N_-^4(\Gamma_1(\infty)).$$

**Reconstruction of  $N^4(X_0^2 \times r) \supset N^4(\Gamma_1(\infty))$ .** We will imitate the same steps as in the RECONSTRUCTION of  $N^4(X_{\text{cubical}}^2)$  above, but now we are GPS, not cubical.

STEP (0). We need, for each  $P \in \{\text{vertex of } X_0^2 \times r\} \in \Gamma(\infty) \times r$ , a pair  $(N^4(P), \{\partial N^4(P) \cap [((X_0^2 \times r) \cup (\Gamma(\infty) \times [r, \beta])) \mid P]\})$ . When the axes  $\xi_0, \zeta$  are disregarded, then the corresponding restrictions will be exactly the ones from Figure 9. If we keep this condition strictly for the  $P \in \Delta^2 \times (\xi_0 = -1)$ , then (22.A) is **automatically satisfied**, provided we proceed correctly along the edges  $[P_1, P_2] \subset \Delta^2 \times (\xi_0 = -1)$  too. To the  $N^4(P)$ 's we add now, for all  $P$ 's, the contribution  $b^3(\beta) \subset B^3(+) \subset \partial N^4(P)$ , and for any  $P \in \Delta^2 \times (\xi_0 = 0)$ , respectively  $P \in \Delta^2 \times (\xi_0 = -1)$ , the contribution  $b^3(-\xi_0)$ , respectively  $b^3(+\xi_0)$ , with  $b^3(\pm \xi_0) \subset B^3(-)$ . We are now at the level of the extended figures à la 9 and, in these, the  $b^3(\pm \xi_0)$ 's are riding on top of the  $\Sigma_\infty^2$ , like the  $b^3(\pm u)$ ,  $u \in \{x, y, z, t\}$ . Of course, in our extended figures we add lines

joining  $b^3(\beta)$  to  $b^3(\pm \xi_0)$  and  $b^3(\beta \text{ or } \pm \xi_0)$  to all the  $b^3$  (space-time). Any crossing of lines  $\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \implies \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array}$  inside  $\partial N^4(P)$  is allowed EXCEPT for the crossing of lines  $b^3(\text{space-time}) \text{ --- } b^3(\text{space-time})$ , belonging to  $\Delta^2 \times (\xi_0 = -1)$ . This way we stay compatible with (22.A), (22.B).

[Repeating myself once, more, compatibility with (22.A) also requires paying attention to what we do along the edges, in particular paying attention to the framings. That should be obvious.] Concerning the allowed crossings notice that they do not change the geometric intersection matrices  $C \cdot h, \eta \cdot B$ . They also leave the topology of  $N^4(\Delta^2 \times (\xi_0 = -1))$  intact.

This ends STEP (0) and next come the STEPS I, II, which are exactly like for the reconstruction of  $N^4(X_{\text{cubical}}^2)$ . This can continue now to a whole reconstruction of  $N^4(2X_0^2)$ , without caring any longer about (22.A), on the  $X_b^2$ -side. Our construction of  $N^2(2X_0^2)$  automatically comes with  $C \cdot h = \text{id} + \text{nil}$ , implying (22.B).

Let us consider now  $N^4(\Gamma_1(\infty)) \subset N^4(2\Gamma(\infty))$ , endowed with the SPLITTING (28.1) and with the {link} from (18.1). But we will restrict our attention to  $N^4(\Gamma_1(\infty))$ , with the splitting restricted to it and with the link [new]  $\subset \partial N^4(\Gamma_1(\infty))$  from (28.2). For each  $P \in X_0^2 \times r$ , the  $\Sigma_\infty^2$  from (27.5) splits  $S^3(P) = \partial N^4(P)$ , via  $S_\infty^2 = \Sigma_\infty^2 \cap S^3(P)$ , into the  $B^3(-)$  and  $B^3(+)$  which we have already met. And, as already mentioned earlier  $S^2(\infty) = S_0^2(P) \times \frac{\beta}{2}$ , with the space of the Figures 9, 24 being  $B^3(-) \cup B^3(\text{in})$  (see Figure 9.bis).

With this, from now on our **link diagram** (27.1) is actually the restriction of a larger link diagram

$$(28.3) \quad \{\text{link}\}[\text{NEW}] \xrightarrow{\Lambda} \Sigma_\infty^2,$$

a generic immersion with prescribed UP/DOWN labels at each crossing. The {link}[NEW] is generated, then, from this LINK DIAGRAM and, next, the canonical framings come from the neighbourhood of (28.3) in  $\Sigma_\infty^2$ .

We move next to the topic of EXTENDED COCORES. Here what I CLAIM is the following:

$$(28.4) \quad \text{Inside } 2X_0^2, \text{ for any point } x \in 2X_0^2 - \sum_{\Gamma_i \subset \Delta^2} \text{int}(D^2(\Gamma_i) \times (\xi_0 = -1)) \text{ there is an}$$

$$\{\text{extended cocore } x\} \subset 2X_0^2,$$

which is a tree based at  $x$ , PROPERLY embedded inside  $2X_0^2$ , cutting transversally the  $2\Gamma(\infty) \subset 2X_0^2$  and which has the feature that at any point  $\{\text{extended cocore } x\} - \{x\}$ , the  $2X_0^2$  is locally split in two. [This means that if  $y \in \{\text{extended cocore } x\} \cap \Gamma(2\infty)$  ( $\neq x$ ), then for each  $2^{\text{d}}$  branch of  $2X_0^2$  at  $e \equiv \{\text{edge of } 2\Gamma(\infty) \text{ containing } y\}$ , there is a  $1^{\text{d}}$  branch of the  $\{\text{extended cocore } x\}$  at  $y$ . We will also say that such a tree like our extended cocore, which locally splits  $2X_0^2$  into two sides, is a **complete tree**.]

Now, we start by considering any  $b_i \in B \cap (\Gamma(1) \times (\xi_0 = -1))$  and show how the all-important  $\{\text{extended cocore } b_i\} \subset 2X_0^2$  is to be constructed. From its root at  $b_i = b_i \times (\xi_0 = -1)$ , the extended cocore anyway starts with the edge  $b_i \times [-1 \leq \xi_0 \leq 0]$ . When we get at  $b_i \times (\xi_0 = 0) \in X_0^2[\text{new}] \mid (\text{truncated}) \equiv X_0^2[\text{new}] - (\Gamma(1) \times (0 > \xi_0 \geq -1)) \cup \Delta^2 \times (\xi_0 = -1)$ , we have two possibilities.

i) (The trivial case) The  $b_i \times (\xi_0 = 0)$  is an “isolated” point of  $X_0^2[\text{new}]$  | (truncated), i.e. it is not touched by the 2-skeleton. [And remember that the  $D^2(\Gamma_i) \times (\xi_0 = 0)$ ’s are NOT part of the 2-skeleton in question.] In this case

$$\{\text{extended cocore } b_i\} \equiv b_i \times [-1 \leq \xi_0 \leq 0].$$

It will turn out that most of the theory in the present paper becomes TRIVIAL in this situation and we will not dwell on it any longer. [When, via the rest of this paper we will understand how to deal with the generic case below, then automatically and moreover trivially so, one will also know how to deal with our trivial case i).]

ii) The **generic** case, when  $b_i \times (\xi_0 = 0) \subset e \subset \Gamma(1) \times (\xi_0 = 0)$  is incident, in  $X^2[\text{NEW}]$  to 2-cells NOT in  $\Delta^2 \times (\xi_0 = 0)$  and then  $\{\text{extended cocore } b_i\}$  is an infinite tree.

In this case, our  $b_i \times (\xi_0 = 0)$  is touched by the infinite collapsing flow unleashed by  $C \cdot h = \text{id} + \text{nilpotent}$  inside  $X_0^2[\text{NEW}]$ .

So, in the generic case

$$\begin{aligned} & \{\text{extended cocore of } b_i \times (\xi_0 = 0)\} = \\ & \bigcup \{\text{all the trajectories of the } 2^d \text{ RED collapsing flow of } 2X_0^2, \text{ hitting } b_i \times (\xi_0 = 0)\}. \end{aligned}$$

In this case, we also have, of course

$$\{\text{extended cocore } b_i \times (\xi_0 = -1)\} = b_i \times [-1 \leq \xi_0 \leq 0] \cup \{\text{extended cocore of } b_i \times (\xi_0 = 0)\}.$$

This definition easily extends to all points  $x \in 2X_0^2$  **not** in  $\bigcup \overset{\circ}{D}^2(\Gamma_i) \times (\xi_0 = -1)$ . Retain that all  $x \in R_1 = \{R_1, R_2, \dots, R_n \subset \Delta^2 \times (\xi_0 = -1)\} + \sum_1^\infty h_n$ , or  $x \in B_1$  or  $x \in \text{int } D^2(C)$ , with  $C$  like in (18.1) have extended cocores, but certainly **not** the  $x \in \text{int } (D^2(\Gamma) \times (\xi_0 = -1))$ .

Even better, let us go now 4-dimensional, then we get a PROPERLY embedded copy of  $B^3 - \{\text{a tame Cantor set of } \partial B^3\}$ , denote it with the same notation as above,

$$\{\text{extended cocore of } x\} \subset N^4 \left( 2\Gamma(\infty) \cup \sum_1^\infty D^2(C_i) \right) \subset N^4(2X_0^2).$$

Here  $x \in \partial\{\text{extended cocore of } x\}$  (conceived now as a  $3^d$  object) and the embedding  $\{\text{exterior cocore } x\} - \{x\} \subset N^4(2X_0^2)$  is proper. [Reminder on the terminology. PROPER means inverse image of compact is compact, while proper means interior to interior and boundary to boundary.]

The  $x \in \text{int } D^2(\Gamma_i)$ ’s DO NOT have such extended cocores.  $\square$

**Compactification.** We have given a general idea how to get our  $N^4(2X_0^2)$ , with some details, like the explicit contributions of the vertices  $b^3(\beta)$ ,  $b^3(\pm \xi_0)$  to be made more explicit later.

But what we have just said above should suffice, for right now. In particular, notice the following decomposition

$$(29) \quad 2X_0^2 = \Delta^2 \text{ (always meaning } \Delta^2 \times (\xi_0 = -1)) \cup \sum_1^K T_i \text{ (= a finite union of trees, each resting with its foot on } \Delta^2) + \sum_1^\infty h_j + \sum_1^\infty D^2(C_j).$$

Here, of course, the  $h_j$ ,  $D^2(C_j)$  are our RED 1-handles and 2-handles, coming with  $C_j \cdot h_i = \text{id} + \text{nil}$  (of the easy type), like in (19). But, for expository purposes, let us be for a while a bit more general, and just

assume, in the context of (29), that we are given a canonical isomorphism between the sets of 1-handles and 2-handles, such that

$$(30) \quad \partial D^2(C_i) \cdot h_j = \delta_{ij} + \{\text{off diagonal } \eta_{ij} \in Z_+\}.$$

We will make it clear, explicitly, when we go from the general (30) to the easy id + nil of (19). But we will stay general, for a short while.

Our  $h_i$  and  $D^2(C_i)$  are actually 4<sup>d</sup> handles of index  $\lambda = 1$  and  $\lambda = 2$ , respectively; we will set for them

$$(31) \quad h_i = (\text{arc } J_i = \text{core of the 1-handle}) \times (3^{\text{d}} \text{ ball } B_i^*, \text{ the cocore}) = J_i \times B_i^*, \quad D^2(C_i) = (\text{disc } D_i = \text{core}) \times (\text{another } 2^{\text{d}} \text{ disc } D_i^*, \text{ the cocore}) = D_i \times D_i^*.$$

With this, the canonical  $\delta_{ij}$  in (30) comes with inclusions:

$$J_i(\text{core of } h_i) \subset \partial D_i^2 = C_i \text{ (we think here in terms of } D^2(C_i) \approx D_i(\text{core})), \partial(B_i^*(\text{cocore})) \supset D_i^*(\text{cocore}).$$

The 4<sup>d</sup> context we talk about here, means the not yet fully explicit  $N^4(2X_0^2)$ . The  $h_i, D^2(C_i)$  are now 4<sup>d</sup> cells with disjointed interiors, out of which we will put together the following connected sum along the respective boundaries

$$h_i \cup D^2(C_i) = h_i \underbrace{\quad \# \quad}_{(\text{core of } h_i) \times (\text{cocore of } D^2(C_i)) = J_i \times D_i^*} D^2(C_i) \underset{\text{DIFF}}{=} B^4 \equiv \text{the “state” } i.$$

Figure 12 should show what we are talking about, with one ambient dimension less.

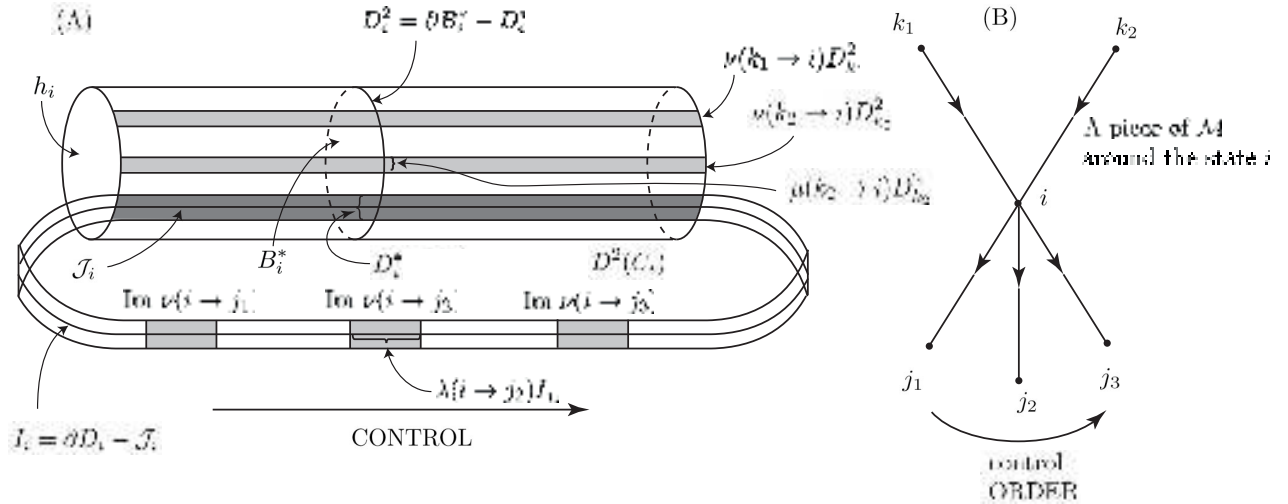


Figure 12.

We display here, with one dimensionless, the state  $i$ .

LEGEND:  $\blacksquare = J_i \times D_i^* \approx \text{Box}(i)$ ;  $\square = \nu(i \rightarrow j_k) \text{Box}(i) \subset \text{Box}(j_k)$ .

$\rightarrow$  = CONTROL arrow. Here, because  $\dim I_i = 1$  there is a natural linear order (well-defined up to a global orientation, which can be chosen arbitrarily, on the set of arrows  $i \rightarrow j$ , which are outgoing from  $i$ . In real life,  $\dim D_i^2 = 2$  and there is NO such order for the arrows incoming into  $i$ .

To the matrix (30) with its leading  $\delta_{ij}$ -term followed by off-diagonal terms, we will associate an oriented graph denoted  $\mathcal{M}$  (30). The vertices of this  $\mathcal{M}$  are the various states  $i$  above, i.e. the  $h_i \cup D^2(C_i)$ 's but another incarnation of  $i$  will be the  $\text{Box}(i)$  defined in (33) below. The off-diagonal part of (30) defines the oriented edges of  $\mathcal{M}$ , via the recipe

$$\# \{\text{edges } i \rightarrow j \text{ in } \mathcal{M}\} = \{\eta_{ij} \text{ in (30)}\}.$$

Such oriented edges will also be called “arrows”.

In connection with the formulae (31) we introduce the following objects, displayed in the Figure 12:

$$(32) \quad \begin{aligned} D_i^2(\text{disc}) &= \partial \text{cocore } h_i - \text{cocore } D^2(C_i) = \partial B_i^* - D_i^* \\ I_i(\text{arc}) &= \partial \text{core } D^2(C_i) - \text{core } h_i = \partial D_i - J_i. \end{aligned}$$

These formulae suggest obvious identifications, rel  $\partial I_i = \partial J_i$ ,  $\partial D_i^2 = \partial D_i^*$ ,

$$(32.1) \quad I_i \approx J_i (= 1^{\text{d}} \text{ core } h_i), \quad D_i^2 \approx D_i^* (= 2^{\text{d}} \text{ cocore of the } 4^{\text{d}} \text{ handle of index } 2, D^2(C_i)).$$

More explicitly,  $D_i^2$  and  $D_i^*$  are the two hemispheres of the 2-sphere  $\partial B_i^*$  and, like wise,  $I_i$  and  $J_i$  the two halves of the circle  $C_i = \partial D_i$ .

I will also introduce now the following purely abstract object

$$(33) \quad \text{Box}(i) \equiv I_i \times D_i^2 = (\partial D^2(C_i) - J_i) \times (\partial B_i^* - D_i^*).$$

The formula (33) can be visualized on Figure 12 and  $\text{Box}(i)$  is a  $3^{\text{d}}$  cell. We say it is “abstract” because it does not come with any a priori canonical embedding into some ambient space.

Of course, in (33)  $I_i, D_i^2$  are like in (32) and there is an abstract identification  $\text{Box}(i) \approx J_i \times D_i^* \subset \partial h_i$ , an embedding suggested by the double shading in Figure 12.

**Lemma 6.** *To each of the  $\eta_{ij}$  arrows  $i \rightarrow j$ , coming with the matrix  $\mathcal{M}$  (30), correspond embeddings, independant from each other,*

$$I_j \xrightarrow{\lambda(i \rightarrow j)} I_i \approx J_i \quad \text{and} \quad D_i^2 \approx D_i^* \xrightarrow{\mu(i \rightarrow j)} D_j^2;$$

see Figure 12.

Out of these two we can extract the following map, which is a smooth embedding when restricted to its domain of definition

$$(34) \quad \text{Box}(i) = I_i \times D_i^2 \supset (\lambda(i \rightarrow j)I_j) \times D_i^2 \xrightarrow{\nu(i \rightarrow j) \equiv \text{id} \times \mu(i \rightarrow j)} I_j \times D_j^2 = \text{Box}(j).$$

Notice here that the  $\lambda(i \rightarrow j)I_j \subset I_i$  is an isomorphic copy of  $I_j$ .

It is best to think of the  $\nu(i \rightarrow j)$  as a purely abstract map. Figures 12 and 13 should help visualizing these things. In the formula (34), for  $x \in I_j$ , the id factor of  $\nu(i \rightarrow j)$ , takes  $\lambda(x) = \lambda(i \rightarrow j)(x)$  back to  $x$ . Whatever minor ambiguity there might be in the definition of the map  $\nu(i \rightarrow j)$ , the point here is that the  $\text{Box}(i)$  goes **cleanly through**  $\text{Box}(j)$ , like in a standard **Markov partition**. This is suggested in the Figure 13.

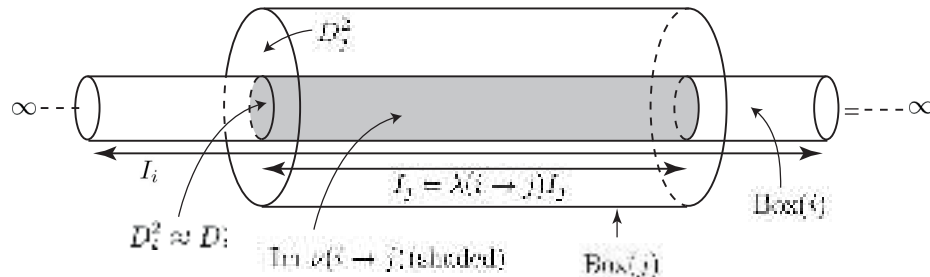


Figure 13.



This figure accompanies the arrow  $i \rightarrow j$  of  $\mathcal{M}$ , and the points  $p \in \text{Box}(i)$  and  $q \in \text{Box}(j)$  which are such that  $q = \nu(i \rightarrow j)p$ , get identified. See here (32.1) too. Our figure shows what the map  $\nu(i \rightarrow j)$  defined in (34) does.

Let  $\mathcal{M} = \sum_{\alpha} \mathcal{M}_{\alpha}$  be the decomposition into connected components and  $A = A(\alpha) \equiv \sum_{i \in \alpha} \text{Box}(i)$ . On the set  $A(\alpha)$  we introduce the equivalence relation  $\mathcal{R} = \mathcal{R}(\alpha)$ , generated as follows: For each arrow  $\{i \rightarrow j\} \in \mathcal{M}_{\alpha}$  and for each  $p \in \{\lambda(i \rightarrow j)I_j \times D_i^2\}$ , the domain of definition of the map  $\nu(i \rightarrow j)$  (34)  $\subset \text{Box}(i)$ , with  $q = \nu(i \rightarrow j)(p) \in \text{Box}(j)$  one has, to begin with,

$$(p, q) \in \mathcal{R}.$$

The whole equivalence relation  $\mathcal{R}$  is, afterwards, generated by these pairs, by adding the obvious things like

$$\{(p, q), (q, r) \in \mathcal{R}\} \implies (p, r) \in \mathcal{R}, \text{ a.s.o.}$$

With this we consider the quotient space

$$(35) \quad X^3(\alpha) \equiv \left( \sum_{i \in \alpha} \text{Box}(i) \right) / \mathcal{R}(\alpha).$$

At the (temporary) level of whole generality, there is a little problem here. In the general case  $\mathcal{M}$  might have oriented ***cycles***. Then, there can be orbits of  $\mathcal{R}$  which, as subsets of  $A(\alpha)$  may fail to be closed sets. Then, continuous functions no longer separate points, and  $X^3(\alpha)$  only makes sense as a ***non-commutative space***, with the algebra of functions which is natural for quotient-spaces, namely matrices of functions, of the form

$$\begin{pmatrix} f(p, p), & f(p, q) \\ f(q, p), & f(q, q) \end{pmatrix},$$

with the matrix algebra multiplication law, i.e. convolution product

$$(f * g)(p, q) = \underbrace{\sum_{(p, r), (r, q) \in R}}_{(p, r), (r, q) \in R} f(p, r) g(r, q) \quad (\text{see [C]}).$$

Fortunately, in our real-life case, when (30) is of the easy id + nil type these complications vanish into thin air. In our real-life case of the easy id + nil matrix (30) we have the following

**Lemma 7.** *When the graph  $\mathcal{M}(30)$  is of the type easy id + nil, then the  $X^3(\alpha)$  from (35) is a mundane connected non-compact 3-manifold with non-empty boundary, which comes naturally equipped with a ***lamination***  $\mathcal{L}_{\alpha}$  by lines.*

The proof of Lemma 7 will be given later. [For the time being I will only give some comments. With any  $\mathcal{M}(30)$  the Markov type Figures 13 are always there. But the issue is what goes on when one puts them together. In the easy id + nilpotent case, they pile up nicely, we have things like (55) below and one gets a smooth final object like in Lemma 7. In particular, with  $\mathcal{M}(30)$  of the type easy nil + nilpotent, there are neither closed orbits nor dangerous going-back trajectories à la  $\text{Wh}^3$ , for  $\mathcal{R}(\alpha)$  and our  $X^3(\alpha)$  from (35) is then a mundane 3-manifold, useful for our present purposes. But let us stop here for one minute and look back. In the most general case, when for some manifold we give a handle-decomposition coming with a canonical identification between the following two sets

$$\{\text{handles of index } \lambda + 1\} \approx \{\text{handles of index } \lambda\},$$

dictated by a corresponding geometric intersection matrix à la (30), then strange objects, quotient spaces which may be non-commutative spaces, in no way directly related with the manifold under investigation, are lurking around.

Of course, with our easy id + nilpotent condition, for the geometric intersection matrix, these additional spaces are no longer **non**-commutative, but mundane manifolds which we will happily use in our constructions. Notice here the following paradigm, the first part of which is a proved result (our Lemma 7) while the second part more like a hunch which deserves, I think, to be investigated:

GSC (i.e. easy id + nilpotent)  $\implies$  an  $X^3(\alpha)$  which is a nice smooth manifold,

and then,

general case (non-GSC)  $\implies$  an  $X^3(\alpha)$  which is a horrible quotient-space possibly

a **non**-commutative space (à la Alain Connes).

It should be stressed that all this little story does not function only in the non-compact context which interests us here, but in the compact case too.] From now on we go back to our real life easy id + nil situation.

In the context of Lemma 7, we denote by  $\bar{\varepsilon}(X^3(\alpha)) \subset X^3(\alpha)$  the closed set of our lamination  $\mathcal{L}_\alpha$ . Transversally to  $\bar{\varepsilon}(X^3(\alpha))$ , the  $X^3(\alpha)$  is a pair

( $R^2$ , a **tame** Cantor set  $\subset R^2$ , i.e. a Cantor set  $C$  such that its inclusion into  $R^2$  factorizes through a smooth line  $C \subset L \subset R^2$ ).

Let us be more specific. Every  $D_i^2$  is a transversal to the lamination  $\mathcal{L}_\alpha$ , the closed set of which is  $\bar{\varepsilon} X^3(\alpha)$  and, we have

$$(\bar{\varepsilon} X^3(\alpha)) \cap D_i^2 = \underbrace{\bigcup_{\substack{i = i_0 \leftarrow i_1 \leftarrow i_2 \leftarrow \dots \\ \text{all the trajectories of } \mathcal{M} \\ \text{ending at } i}}}_{\text{all the trajectories of } \mathcal{M} \text{ ending at } i} \bigcap_{j=1}^{\infty} \mu(i_j \rightarrow i) D_{i_j}^2.$$

Let's go now to the smooth 4<sup>d</sup> version of (29),

$$(36) \quad N^4(2X_0^2) = \left( N^4(2\Gamma(\infty)) - \sum_1^{\infty} h_i \right) + \sum_1^{\bar{n}} D^2(\Gamma_j) + \sum_1^{\infty} (h_i \cup D^2(C_i)) = N^4(\Delta^2) \cup \sum_{\ell=1}^K \{ \text{the 4<sup>d</sup> regular neighbourhood of a tree } T_\ell \text{ resting on } \partial N^4(\Delta^2), \text{ and which we continue to denote by } T_\ell \} + \sum_1^{\infty} h_i + \sum_1^{\infty} D^2(C_i).$$

Here  $N^4(2\Gamma(\alpha)) - \sum_1^{\infty} h_i \equiv \left\{ \text{the } N^4(2\Gamma(\infty)) \text{ split along } \sum_1^{\infty} \text{cocore } h_i \right\} \supset N^4(\Gamma(1)) \supset \sum_1^n R_i$  (the RED 1-handles if  $N^4(\Delta^2)$ ).

Let  $\varepsilon(T_\ell)$  denote the set of end-points of the tree  $T_\ell$ . At 4<sup>d</sup> level we have an obvious smooth compatification, albeit a completely trivial one

$$(37) \quad T_\ell^\wedge \equiv \{ \text{The 4<sup>d</sup> } T_\ell \} \cup \varepsilon(T_\ell) \equiv B^4(T_\ell) \underset{\text{DIFF}}{=} B^4, \text{ with } \varepsilon(T_\ell) \subset \{ \text{a tame Cantor set inside } \partial B^4(T_\ell) \}.$$

Of course,  $\varepsilon(T_\ell)$  lives at the infinity of  $T_\ell$ .

So, we get here another diffeomorphic model for  $N^4(\Delta^2)$

$$(37.1) \quad N^4(\Delta^2)^\bullet = N^4(\Delta^2) \cup \sum_{\ell=1}^K \hat{T}_\ell = \left( N^4(2\Gamma(\infty)) - \sum_1^{\infty} h_n \right)^\wedge.$$

It is essential here that we have  $\left(\sum_1^{\bar{n}} \Gamma_j\right) \cap h_i = \emptyset$ , a feature which will be lost when, for the so-called BALANCING purpose, we will have to extend the  $\sum_1^{\bar{n}} \Gamma_j$ .

For each  $\alpha$  in  $\mathcal{M} = \sum_{\alpha} \mathcal{M}_{\alpha}$ , the following object is a smooth non-compact 4-manifold with non-empty boundary

$$(38) \quad \text{LAVA}_{\alpha} \equiv \sum_{i \in \alpha} (h_i \cup D^2(C_i)),$$

and with this we also set

$$\text{LAVA} \equiv \sum_{\alpha} \text{LAVA}_{\alpha}.$$

This LAVA may well touch the  $\sum_1^n R_i \times (\xi_0 = -1)$  but not to  $\sum_1^{\bar{n}} D^2(\Gamma_j)$ . We also have

$$(38.1) \quad \delta \text{LAVA}_{\alpha} \equiv (\partial \text{LAVA}_{\alpha}) \cap \partial \left( N^4(2X_0^2) - \sum_1^{\infty} h_k \right), \quad \delta \text{LAVA} \equiv \sum_{\alpha} \delta \text{LAVA}_{\alpha}.$$

Here is how  $N^4(2X_0^2)$  is structured

$$(39) \quad N^4(2X_0^2) = \left[ \left( N^4(2\Gamma(\infty)) - \sum_1^{\infty} h_k \right) \underbrace{\cup}_{\delta \text{LAVA}} \text{LAVA} \right] + \sum_1^{\bar{n}} D^2(\Gamma_j).$$

In the RHS of this formula, LAVA is a PROPER codimension zero submanifold. Each component  $\text{LAVA}_{\alpha} \subset \text{LAVA}$  is actually itself PROPER, and it is split from the rest by the connected codimension one submanifold  $\delta \text{LAVA}_{\alpha}$ . Very importantly, in the context of (39), the 2-handles  $D^2(\Gamma_j)$  are directly attached to  $N^4(2\Gamma(\infty)) - \sum_1^{\infty} h_k$ .

**Lemma 8.** 1) *As far as the ends are concerned, we have*

$$\varepsilon \left( N^4(2\Gamma(\infty)) - \sum_1^{\infty} h_k \right) = \sum_1^K \varepsilon(T_i),$$

*a formula which would of course no longer be true with the LHS replaced by  $\varepsilon(N^4(2\Gamma(\infty)))$ , (in which case we would only find a quotient space projection  $\sum_1^K \varepsilon(T_i) \twoheadrightarrow \varepsilon(N^4(2\Gamma(\infty)))$ ). Also, there is an easy smooth compactification*

$$(40) \quad \left( N^4(2\Gamma(\infty)) - \sum_1^{\infty} h_k \right)^{\wedge} = \left( N^4(2\Gamma(\infty)) - \sum_1^{\infty} h_k \right) \cup \varepsilon \left( N^4(2\Gamma(\infty)) - \sum_1^{\infty} h_k \right) \stackrel{=}{\text{DIF}} \\ \stackrel{=}{\text{DIF}} n \# (S^1 \times B^3) \quad (\text{with } n = \# R \cap \Gamma(1)).$$

*Moreover, in the same vein, we have the diffeomorphism*

$$(40.1) \quad N^4(\Delta^2) \stackrel{=}{\text{DIF}} \left( N^4(2\Gamma(\infty)) - \sum_1^{\infty} h_k \right)^{\wedge} + \sum_1^{\bar{n}} D^2(\Gamma_j),$$

with the RHS of (40.1) being the same object as the  $N^4(\Delta^2)^\bullet$  above (see (37.1)).

1-bis) Without any loss of generality,  $\left(N^4(2\Gamma(\infty)) - \sum_1^\infty h_k\right)^\wedge$  is diffeomorphic to the closure of  $N^4(2\Gamma(\infty)) - \sum_1^\infty h_k$  in the ambient space  $N^4(2X_0^2)$ ; so we have the following diffeomorphism between the abstract compactification occurring in the LHS of the formula below, and the mundane closure inside the ambient space  $N^4(2X_0^2)$  (39)

$$\left(N^4(2\Gamma(\infty)) - \sum_1^\infty h_n\right)^\wedge \stackrel{\text{DIF}}{=} \overline{\left(N^4(2\Gamma(\infty)) - \sum_1^\infty h_n\right)}.$$

This kind of equalities abund in the sequel of this paper, but we will not always bother to write them down explicitly. The reader should not have any difficulty in guessing them.

The  $\varepsilon\left(N^4(2\Gamma(\infty)) - \sum_1^\infty h_k\right) \subset \partial\left(N^4(2\Gamma(\infty)) - \sum_1^\infty h_k\right)^\wedge$  is a tame Cantor set and we also have the following equality between spaces of ends  $\varepsilon\left(N^4(2\Gamma(\infty)) - \sum_1^\infty h_k\right) = \varepsilon\left(\partial\left(N^4(2\Gamma(\infty)) - \sum_1^\infty h_k\right)\right)$ .

2) There is a **PROPER smooth embedding**

$$(41) \quad \sum_\alpha X^3(\alpha) \xrightarrow{J} \partial\left(N^4(2\Gamma(\infty)) - \sum_1^\infty h_k\right) = \partial\left(N^4(2\Gamma(\infty)) - \sum_1^\infty h_k\right)^\wedge - \varepsilon\left(\partial\left(N^4 - \sum_1^\infty h_k\right)\right).$$

Moreover, in terms of (38.1) we have here the following equality of sets

$$(42) \quad \delta \text{LAVA}_\alpha = JX^3(\alpha) \approx X^3(\alpha).$$

3) For each  $\alpha$  there is a diffeomorphism of pairs

$$(43) \quad (\text{LAVA}_\alpha, \delta \text{LAVA}_\alpha) \stackrel{\text{DIF}}{=} (X^3(\alpha) \times [0, 1] - \left(\underbrace{\bar{\varepsilon}(X^3(\alpha))}_{\text{the closed set of the lamination } \mathcal{L}_\alpha} \times \{1\}\right), X^3(\alpha) \times \{0\}).$$

The proof and various embellishments of this lemma, will follow in this section, later on. We introduce now the endpoint compactification

$$(\delta \text{LAVA}_\alpha)^\wedge \equiv \delta \text{LAVA}_\alpha \cup \varepsilon(\delta \text{LAVA}_\alpha).$$

As a consequence of (42), (43), we have an (almost) SMOOTH COMPACTIFICATION OF LAVA

$$(44) \quad \text{LAVA}_\alpha^\wedge = (\delta \text{LAVA}_\alpha \times [0, 1]) \underbrace{\cup}_{\delta \text{LAVA}_\alpha} (\delta \text{LAVA}_\alpha)^\wedge = (\delta \text{LAVA}_\alpha \times [0, 1]) \cup \varepsilon(\text{LAVA}_\alpha),$$

and here the  $\delta \text{LAVA}_\alpha \times [0, 1] \subset \text{LAVA}_\alpha^\wedge$  is a bona fide smooth manifold. But it should be stressed here that (44) as well as the

$$(\delta \text{LAVA}_\alpha)^\wedge \equiv \delta \text{LAVA}_\alpha \cup \varepsilon(\delta \text{LAVA}_\alpha)$$

are provisional formulae, soon to be superseded by more appropriate and more accurate versions. Also, (43), (44), ... are ABSTRACT, meaning that they come, a priori, without any connections with the embedding  $\text{LAVA}_\alpha \subset N^4(2X_0^2)$ ; all this will be corrected in time.

In the context of (43), the induces diffeomorphism

$$\delta \text{LAVA}_\alpha = X^3(\alpha) \times \{0\},$$

is nothing else but the following equality among subsets of  $\partial \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_n \right)$ , the (42), which I will re-write here

$$\delta \text{LAVA}_\alpha = JX^3(\alpha) \approx X^3(\alpha).$$

One should not mix up the  $\varepsilon(\delta \text{LAVA}_\alpha) \equiv \{\text{the space of ends in the sense of Freudenthal, Hopf et al}\}$  and the  $\varepsilon(\text{LAVA}_\alpha)$  which will NOT mean ends, but the set of points at infinity of  $\text{LAVA}_\alpha$ , to be made explicit below, and which is certainly not totally discontinuous, since the lamination  $\mathcal{L}_\alpha$  contributes to  $\varepsilon(\text{LAVA}_\alpha)$ . Each leaf of it joins a pair of points at infinity, living in  $\varepsilon(\delta \text{LAVA}_\alpha)$ . With this

$$(44.1) \quad \varepsilon(\text{LAVA}_\alpha) = \bar{\varepsilon}(X^3(\alpha)) \cup \varepsilon(\delta \text{LAVA}_\alpha) \text{ (and this is a set of ends).}$$

The embedding  $J$  from (41) induces, automatically, a continuous map, at the level of the ends à la Hopf-Freudenthal

$$(45) \quad \sum_\alpha \varepsilon(X^3(\alpha)) \xrightarrow{E} \varepsilon \left( N^4 - \sum_1^\infty h_k \right) = \varepsilon \left( \partial \left( N^4 - \sum_1^\infty h_k \right) \right).$$

For each  $\alpha$  we consider the closed subspace  $\lambda_\alpha \equiv E(\varepsilon(X^3(\alpha))) \subset \varepsilon \left( N^4 - \sum_1^\infty h_k \right)$  and the quotient-space projection

$$(45.1) \quad \begin{array}{c} \varepsilon(\delta \text{LAVA}_\alpha) = \varepsilon(X^3(\alpha)) \twoheadrightarrow \lambda_\alpha. \\ \uparrow \qquad \qquad \qquad \uparrow \\ \approx \text{ (see (42))} \end{array}$$

The compact space  $\lambda_\alpha$  are not necessarily disjointed and we have

$$(46) \quad \overline{\text{Im } E} = \overline{\bigcup_\alpha \lambda_\alpha} = \bigcup_\alpha \lambda_\alpha \cup \lambda_\infty,$$

where  $\lambda_\infty \subset \varepsilon \left( N^4 - \sum_1^\infty h_k \right)$  is the closed set of points  $p_\alpha$  such that we can find a sequence  $p_\alpha \in \lambda_\alpha$  with  $\lim_{\alpha=\infty} p_\alpha = p_\infty$ .

We will sharpen now the definition of  $\text{LAVA}_\alpha^\wedge$  given in (44) by setting, from now on the following improved (and/or corrected) form of (44), namely

$$(47) \quad \text{LAVA}_\alpha^\wedge \equiv (\delta \text{LAVA}_\alpha \times [0, 1]) \cup \lambda_\alpha.$$

Notice that the RHS of this formula, which supersedes (44) is a quotient-space of the RHS of (44). Similarly, we will set

$$(47.1) \quad (\delta \text{LAVA}_\alpha)^\wedge \equiv (\delta \text{LAVA}_\alpha) \cup \lambda_\alpha.$$

So, from now on, we use the sharper  $\lambda_\alpha$  instead of the ABSTRACT  $\varepsilon(\delta \text{LAVA}_\alpha)$ .

**The compactification Lemma 9.** 1) *Via a smooth, not necessarily ambient isotopy of the  $N^4(\Delta^2) \cup \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_k \right)$ , inside the ambient space  $N^4(2X_0^2)$ , we can get, for each  $\alpha$ , the following equality, connecting the LHS CONCRETE pair to the RHS ABSTRACT pair, as conceived from now on*

$$(48) \quad (\overline{\text{LAVA}_\alpha}, \overline{\delta \text{LAVA}_\alpha}) = \left( \underbrace{(\delta \text{LAVA}_\alpha \times [0, 1]) \cup \lambda_\alpha}_{\text{LAVA}_\alpha^\wedge}, \underbrace{(\delta \text{LAVA}_\alpha \cup \lambda_\alpha) \times \{0\}}_{(\delta \text{LAVA}_\alpha)^\wedge} \right).$$

2) The smooth non-compact 4-manifold, part of  $N^4(2X_0^2)$

$$\left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_k \right) \underbrace{\bigcup_{\delta \text{LAVA}}}_{\text{LAVA}} \quad (\text{see (39)})$$

has the following canonical smooth compactification

$$(49) \quad \left[ \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_k \right) \cup \text{LAVA} \right]^\wedge \equiv \left\{ \left\{ \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_k \right)^\wedge \cup \bigcup_\alpha \text{LAVA}_\alpha^\wedge, \text{ where each } \lambda_\alpha \subset \text{LAVA}_\alpha^\wedge \text{ is identified with its counterpart in } \varepsilon \left( \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_k \right)^\wedge \right) \right\} \right\}^\wedge = \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_k \right)^\wedge \cup \underbrace{\bigcup_\alpha \left( \text{LAVA}_\alpha^\wedge \cup \lambda_\alpha \right)}.$$

This is now a closed subset of  $\left( \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_k \right) \cup \text{LAVA} \right)^\wedge$

Here the various  $\delta \text{LAVA}_\alpha \times [0, 1] \subset \text{LAVA}_\alpha^\wedge$  are 2-by-2 disjointed, and also  $\text{LAVA}_\alpha \subset \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_k \right) \cup \text{LAVA}$  is a PROPER submanifold, unlike the  $\delta \text{LAVA}_\alpha \times [0, 1] \subset \left( \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_k \right) \cup \text{LAVA} \right)^\wedge$ , where the LHS lacks the  $\lambda_\alpha$ .

In the same vein, the  $\bigcup_\alpha \text{LAVA}_\alpha^\wedge \subset \left( \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_k \right) \cup \text{LAVA} \right)^\wedge$  is NOT a closed subset, since the

$$\lambda_\infty \subset \varepsilon \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_k \right)$$

is still lacking to it. We have to add the  $\lambda_\infty$ , like in (49) in order to clinch the closure. We have now a diffeomorphism

$$(50) \quad \left[ \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_k \right) \cup \text{LAVA} \right]^\wedge \stackrel{\text{DIFF}}{=} \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_k \right)^\wedge \stackrel{\text{DIFF}}{=} n \# (S^1 \times B^3).$$

3) Finally, there is a diffeomorphism, giving our model for  $N^4(\Delta^2)$ , from now on, at least until further notice,

$$(51) \quad \boxed{N^4(2X_0^2)^\wedge = \left[ \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_k \right) \cup \text{LAVA} \right]^\wedge + \sum_1^{\bar{n}} D^2(\Gamma_j) \stackrel{\text{DIFF}}{=} N^4(\Delta^2)}.$$

**Reminder.** In this formula, the 2-handles  $D^2(\Gamma_j)$  are attached directly to  $N^4(2\Gamma(\infty)) - \sum_1^\infty h_n$ . We have

$$(51.1) \quad \left\{ \left[ \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_n \right) \cup \text{LAVA} \right]^\wedge + \sum_1^{\bar{n}} D^2(\Gamma_i) \right\} - \left\{ \left[ \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_n \right) \cup \text{LAVA} \right] + \sum_1^{\bar{n}} D^2(\Gamma_i) \right\} \\ = \left\{ \varepsilon \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_n \right) + \sum_\alpha \varepsilon(\text{LAVA}_\alpha) \text{ (44.1)} \right\};$$

here, in the LHS we see twice the same expression, first with a hat, then without one. Clearly also, the  $\sum D^2(\Gamma_i)$  can be erased from that LHS. Finally, the RHS should be read like in the (51.2) below.

Let us make now explicit the topology of the disjointed union occurring in the RHS of (51.1). We start by adding, to each  $\bar{\varepsilon}(X^3(\alpha))$ , the set of endpoints, living at its infinity, i.e. the

$$\varepsilon(\bar{\varepsilon} X^3(\alpha)) = \varepsilon X^3(\alpha) \subset \varepsilon(\text{LAVA}_\alpha).$$

This is glued then to  $\varepsilon\left(N^4(2\Gamma(\infty)) - \sum_1^\infty h_n\right)$  according to the surjective map from (45.1)

$$\varepsilon(X^3(\alpha)) \twoheadrightarrow \lambda_\alpha \subset \varepsilon\left(N^4(2\Gamma(\infty)) - \sum_1^\infty h_n\right).$$

This *is* the topology which (51.1) inherits from the ambient space. And then, in the same spirit as in formulae (48) and (49), we can express now the space in (51.1) as follows, and in the formula below, each  $\lambda_\alpha$  occurring in  $\bar{\varepsilon}(X^3(\alpha)) \cup \lambda_\alpha$  is to be identified with its counterpart  $\lambda_\alpha \subset \varepsilon\left(N^4(2\Gamma(\infty)) - \sum_1^\infty h_n\right)$ :

$$(51.2) \quad \varepsilon\left(N^4(2\Gamma(\infty)) - \sum_1^\infty h_k\right) \cup \sum_\alpha (\bar{\varepsilon}(X^3(\alpha)) \cup \lambda_\alpha).$$

This is exactly what lives at the infinity of the space  $\left[\left(N^4(2\Gamma(\infty)) - \sum_1^\infty h_k\right) \cup \text{LAVA}\right]^\wedge$ , i.e. it is exactly

$$\begin{aligned} & \left[\left(N^4(2\Gamma(\infty)) - \sum_1^\infty h_k\right) \cup \text{LAVA}\right]^\wedge - \left[\left(N^4(2\Gamma(\infty)) - \sum_1^\infty h_k\right) \cup \text{LAVA}\right] = \\ & \overline{\left[\left(N^4(2\Gamma(\infty)) - \sum_1^\infty h_k\right) \cup \text{LAVA}\right]} - \left[\left(N^4(2\Gamma(\infty)) - \sum_1^\infty h_k\right) \cup \text{LAVA}\right]. \end{aligned}$$

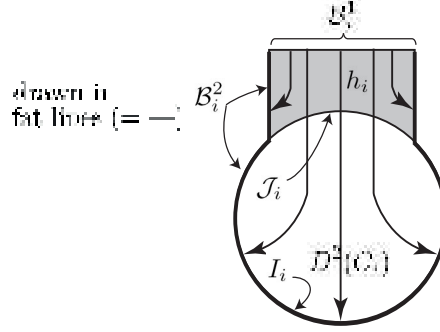
The various  $\lambda_\alpha \subset \varepsilon\left(N^4(2\Gamma(\infty)) - \sum_1^\infty h_k\right)$  are, generally speaking, not disjointed from each other. Moreover, we also find  $\lambda_\infty \subset \varepsilon\left(N^4(2\Gamma(\infty)) - \sum_1^\infty h_n\right)$ .

**The proofs of Lemmas 7, 8 and 9.** On the boundary of the 4-ball state  $(i) = h_i \cup D^2(C_i)$ , we will consider the following two pieces

$$(52) \quad \mathcal{B}_i^1 \equiv J_i \times D_i^2 \longleftrightarrow \partial\{\text{state } i\} \longleftrightarrow \mathcal{B}_i^2 \equiv (\partial J_i \times B_i^*) \cup (I_i \times D_i^*),$$

coming with  $\mathcal{B}_i^1 \cap \mathcal{B}_i^2 = \partial J_i \times D_i^2 \subset \partial J_i \times B_i^*$ , see here (32). Here, of course  $\dim(\text{state}(i)) = 4$ ,  $\dim \mathcal{B}_i^1 = \dim \mathcal{B}_i^2 = 3$ ,  $\dim(\mathcal{B}_i^1 \cap \mathcal{B}_i^2) = 2$ . Moreover

$$\partial(\text{state}(i)) = \mathcal{B}_i^1 \cup \mathcal{B}_i^2 \cup \{D_i \times \partial D_i^*, \text{ the lateral surface of the 2-handle}\}.$$



**Figure 14.**

We illustrate here, with two dimension less, the formula (52). The arrows are in red.

With red lines, we have suggested the **flattening retraction**

$$\begin{array}{ccc} \mathcal{B}_i^1 \subset \{\text{state } i\} & \twoheadrightarrow & \mathcal{B}_i^2 \\ | & & \uparrow \\ & \approx & \end{array}$$

This same figure can be read as a profile view of the whole  $\{\text{state } i\} = h_i \cup D^2(C_i)$ , with  $J_i$  and  $\partial D_i = I_i \cup J_i$  displayed, and with profile views of the  $\mathcal{B}_i^1, \mathcal{B}_i^2$  too.

With one dimension less, formula (52) can be visualized in Figure 14.

Notice that the dumbbell-like connected 3-manifold from (52) and from the Figure 14, the

$$\mathcal{B}_i^2 = \{\text{attaching zone of } h_i\} \cup \{(\text{attaching zone of } D^2(C_i)) - (\text{lateral surface of } h_i)\},$$

is exactly that part of  $\partial(\text{state } i)$  via which, in the absence of incoming arrows  $\{k \rightarrow i\} \in \mathcal{M}$ , our 4-cell state  $(i)$  is SPLIT from the rest of the world. And then, those arrows  $k \rightarrow i$  hit our state  $(i)$  via the piece  $\mathcal{B}_i^1$ .

There exists a FLATTENING RETRACTION suggested in the Figure 14

$$(53) \quad \begin{array}{ccc} \mathcal{B}_i^1 \hookrightarrow \{\text{state } i\} & \xrightarrow{F=F(i)} & \mathcal{B}_i^2 \\ | & & \uparrow \\ & \approx & \\ & F|_{\mathcal{B}_i^1} = \text{diffeomorphism} & \end{array}$$

Then, there is also a canonical isomorphism (see (32.1))

$$(54) \quad \mathcal{B}_i^1 = J_i \times D_i^2 \xrightarrow[\approx]{\eta_i} I_i \times D_i^2 = \text{Box}(i) \quad (\text{see (33)})$$

and so, via (53), (54),  $\mathcal{B}_i^1, \mathcal{B}_i^2$  and  $\text{Box}(i)$  are three distinct diffeomorphic models of the same objet and, in what will follow next we will happily and freely move from one of these models to another, without bothering to change the notation. In this context, in our present easy id + nilpotent context, it can always be assumed that

$$(55) \quad \lim_{n \rightarrow \infty} \text{length } J_n = \infty, \quad \lim_{n \rightarrow \infty} \text{diam } D_n^n = 0, \quad \text{for } n \rightarrow \infty \text{ in } \mathcal{M}.$$



It should be stressed that the easy id + nilpotent condition is necessary at this point. Once that condition is satisfied, we can impose consistently the **metric** requirement that when in the RED matrix  $C \cdot h$  there is an arrow  $i \rightarrow j$ , then we also have that

$$\text{length } I_i > \text{length } I_j \quad \text{and} \quad \text{diameter } D_i^2 < \text{diameter } D_j^2,$$

like in the Figure 13. Starting from that, it is not hard to impose the condition (55) too.

Without our easy id + nil, like for instance in the case of the classical Whitehead manifold  $\text{Wh}^3$ , these things are not possible. The difficult id + nil does not allow such nice metric conditions. It should be an amusing exercise to figure out the (35) for the Whitehead manifold. With these things, let us go now to Lemma 7, where we fix a Box, the  $\text{Box}(i) = I_i \times D_i^2$ , and consider all the infinite trajectories of our oriented graph  $\mathcal{M}$  (easy id + nil) incoming into  $i$ . They have the general form below

$$(56) \quad i \equiv i_0 \leftarrow i_1 \leftarrow i_2 \leftarrow i_3 \leftarrow \dots$$

With this, we introduce now

$$\bar{\varepsilon} X^3(\alpha) \mid \text{Box}(i) \equiv \underbrace{\sum_{i = i_0 \leftarrow i_1 \leftarrow i_2 \leftarrow \dots}}_{i_n} \bigcap_{i_n} \nu(i_n \rightarrow i)(\lambda(i_n \rightarrow i) I_i \times D_{i_n}^2),$$

and this formula should make the transversal structure of the lamination  $\mathcal{L}_\alpha$  transparent. Next, of course, the lamination itself is defined by

$$\bar{\varepsilon} X^3(\alpha) = \bigcup_i (\bar{\varepsilon} X^3(\alpha) \mid \text{Box}(i)).$$

Our lamination is without holonomy, since  $\mathcal{M}$  has no closed orbits.

**Remark.** The easy id + nil implies that our states  $i$  may be labelled by positive integers  $n \in \mathbb{Z}_+$ , s.t. if there is an arrow  $i_n \rightarrow i_m$  in  $\mathcal{M}$ , then  $n > m$ . Of course, also, our oriented graph  $\mathcal{M}$  is not necessarily a tree, but it certainly has no closed ORIENTED orbits. When the big  $\mathcal{M} = \sum_{\infty} \mathcal{M}_\alpha$  is being considered, then we may identify

$$\{\alpha\} \cong \{\text{the set of FINAL states, i.e. the states which have no outgoing arrows}\}.$$

We go back now to a general situation when for our  $\mathcal{M}$  we have chosen an order on the states  $i_\alpha$ , s.t. when there is an arrow  $i_\alpha \rightarrow i_\beta$  then  $i_\alpha > i_\beta$  (or, schematically,  $\alpha > \beta$ ). We have

$$(57) \quad \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_k \right) \cup \text{LAVA} = \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_k \right) \cup \bigcup_{i_n} \{\text{state } i_n\} \\ \supset \sum_{i_n} \mathcal{B}_{i_n}^1, \text{ with this inclusion map being PROPER.}$$

[The  $\mathcal{B}_j^1$ 's are well touched by the higher  $\mathcal{B}_i^2$ 's, but inside our  $\left( N^4(2\Gamma(\infty)) - \bigcup_1^\infty h_n \right) \cup \text{LAVA}$ , they are 2-by-2 disjointed. This gives us an injective map, occurring in (57), call it

$$(57.1) \quad \sum_{i_n} \mathcal{B}_{i_n}^1 \xrightarrow{j} \text{LAVA.}]$$

Here we have

$$(58) \quad \underbrace{\left[ \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_k \right) \cup \bigcup_{n=1}^{N-1} \{\text{state } i_n\} \right]}_{\text{we call this } X^4(N-1)} \cap \{\text{state } i_N\} = \mathcal{B}_{i_N}^2,$$

and

$$\left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_k \right) \cap \{\text{state } i_1\} = \mathcal{B}_{i_1}^2.$$

We will denote

$$\nu(i_N \rightarrow) \equiv \sum_{\text{all } j < i_N} \{\text{the source } \lambda(i_N \rightarrow j) I_j \times D_{i_N}^2 \text{ (see Figure 13) of the map } \nu(i_N \rightarrow j) \text{ (34)}\} \subset \mathcal{B}_{i_N}^2.$$

Here, inside our  $\sum_{j < i_N}$  things are not disjointed, but we will not make this explicit.

When we consider the decomposition

$$\mathcal{B}_{i_N}^2 = (\mathcal{B}_{i_N}^2 - \nu(i_N \rightarrow)) \cup \nu(i_N \rightarrow)$$

then, in terms of (58), the  $\mathcal{B}_{i_N}^2 - \nu(i_N \rightarrow)$  goes to  $\partial \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_k \right)$ , while the  $(\nu_{i_N} \rightarrow)$  goes to the  $\left\{ \text{still free part of } \sum_1^{N-1} \mathcal{B}_n^1 \right\} \subset \partial X^4(N-1)$ .

The following map, defined by composing the  $F(i)$ 's from (53)

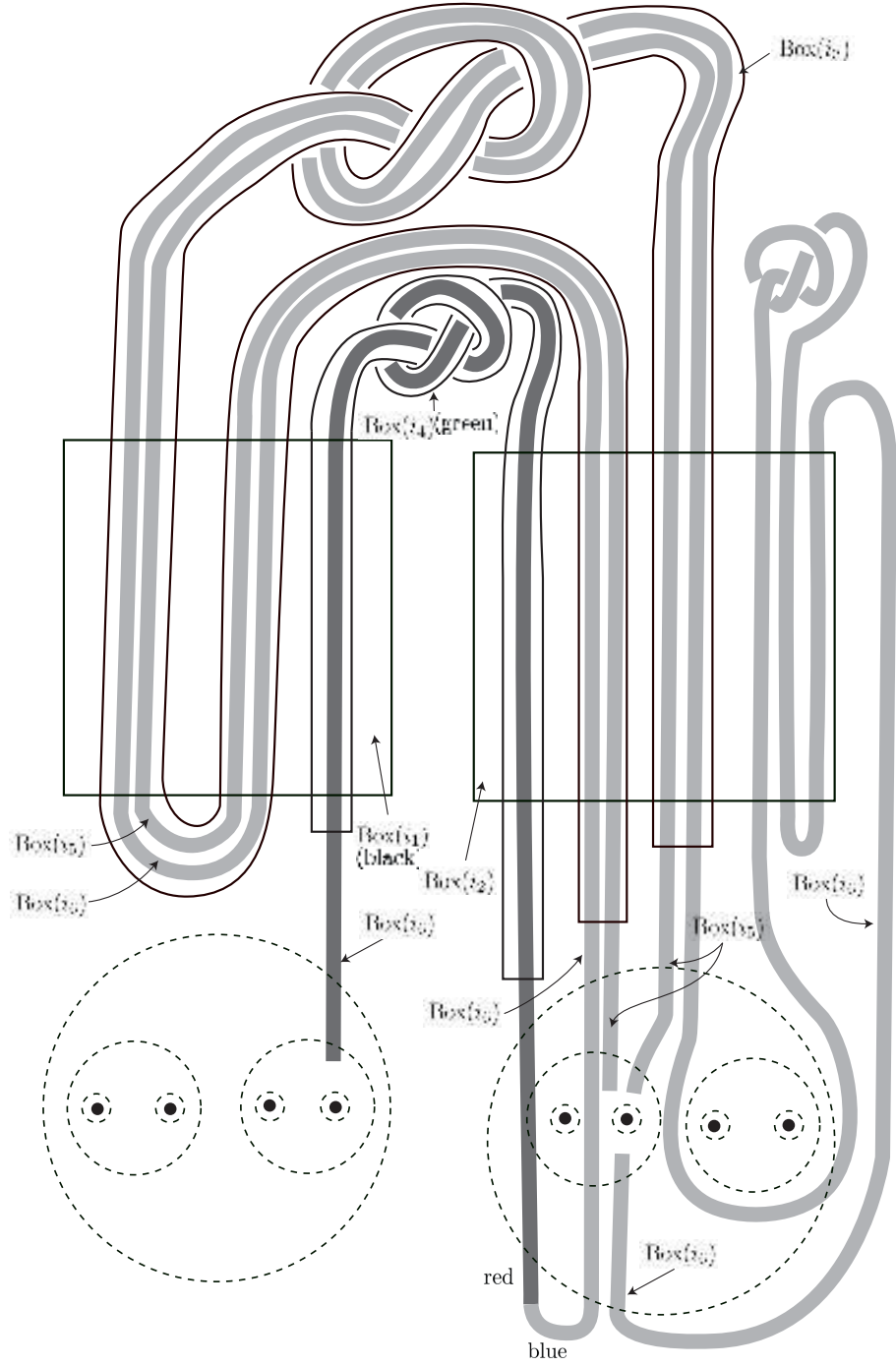
$$(59) \quad \{\text{state } i_N\} \xrightarrow{\Phi(i_N) \equiv F(i_1) \circ F(i_2) \circ \dots \circ F(i_N)} \partial \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_n \right),$$

is well-defined on  $\mathcal{B}_{i_N}^2 \subset \{\text{state } i_N\}$  which it finally flattens on  $\partial \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_n \right)$ . Moreover, when we put together the various  $\Phi(i_N)$ 's then we get the following big, PROPER **flattening map**

$$(60) \quad \text{LAVA} \equiv \bigcup_N \{\text{state } i_N\} \xrightarrow{\Phi(\infty) \equiv \bigcup_N \Phi(i_N)} \partial \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_n \right).$$

When  $j < N$ , then  $\Phi(i_N)$  factorizes through  $\Phi(i_j)$ , and hence the two are compatible.

The big  $\Phi(\infty)$  is a retraction of LAVA onto the  $\delta \text{LAVA} \subset \partial \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_n \right)$ , and we have  $\Phi(\infty) \text{LAVA} = \delta \text{LAVA}$ .



**Figure 15.**

This figure should illustrate the PROPER embedding  $J$  from (41).

The plane of our figure is supposed to be

$$\partial \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_n \right).$$

The fat points ( $= \bullet$ ) are in  $\varepsilon \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_n \right)$ . One should notice that our  $\mathcal{M}$  is generally speaking, NOT a tree, it can have closed non-ordered cycles (not orbits or trajectories). Notice here the occurrence of the following closed curve, contained in  $\mathcal{M}$ , but which is NOT a cycle

$$\begin{array}{ccc} i_6 & \longrightarrow & i_4 \\ \downarrow & & \downarrow \\ i_3 & \longrightarrow & i_1 \end{array}$$

and the graph becomes more complicated when we introduce the  $i_2, i_5$  too.

LEGEND: BLACK =  $i_1$  and  $i_3$ , GREEN =  $i_4$ , RED and BLUE =  $i_6$ .

We have a commutative diagram

$$(61) \quad \begin{array}{ccc} \text{LAVA} & \xleftarrow{j \text{ (57.1)}} & \sum_{i_n} \mathcal{B}_{i_n}^1 \cong \sum_n \text{Box}(i_n) \\ & \searrow \Phi(\infty) & \swarrow \text{canonical quotient map projection; and here, see (35).} \\ & & \sum_{\alpha} X^3(\alpha) \\ & \searrow J \text{ (our PROPER map } J \text{ (41))} & \\ & & \partial \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_k \right) \end{array}$$

and here  $\text{Im } \Phi(\infty) = \text{Im } J = \delta \text{LAVA}$ .

Figure 15 should give an impressionistic idea of what the map  $J$  may look like. Inside each given Box we only see many parallel strands, telescopically embedded inside each other according to the prescription (see the beginning of Lemma 6)

$$D_\ell^2 \xrightarrow{\mu(\ell \rightarrow m)} D_m^2,$$

which is an embedding. Of course, the  $J$ -image of each individual box is no longer multilinear, like it is in the abstract model.

Very importantly, **no knotting or linking** ever occurs, inside any given individual box. But, generally speaking, the global  $X^3(\alpha)$  is highly non-simply-connected and knotted too. This  $\pi_1 \neq 0$  issue stems from the fact that  $\mathcal{M}$  is not necessarily a tree and also from the multiplicities of the given contacts  $i \rightarrow j$ .

Finally, inside the ambient space  $\partial \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_k \right)$  the  $J \sum_{\alpha} X^3(\alpha) \approx \sum_{\alpha} X^3(\alpha)$  is, generally speaking, highly knotted and linked. All these things having been said, one proves (43) by letting each  $\text{LAVA}_{\alpha}$  grow naturally and very slowly, so as to respect our various smoothness conditions, out of its seed  $JX^3(\alpha) \approx X^3(\alpha)$ . The closer it comes to  $\varepsilon \left( \partial N^4(2\Gamma(\infty)) - \sum_1^\infty h_n \right)$  the slower LAVA is supposed to grow and, also, the height to which it grows is becoming smaller and smaller. This is suggested in the Figure 16.

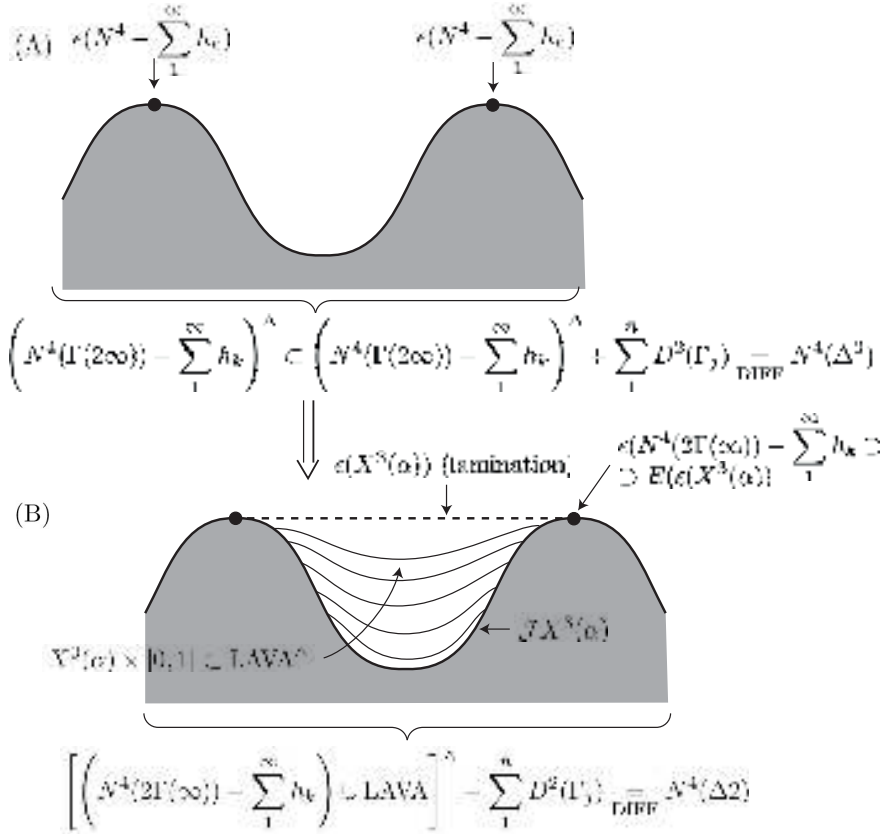
The growth process was suggested here as it is happening inside the ambient space  $N_1^4(2X_0^2)^\wedge = N^4(2X_0^2)^\wedge \cup (\partial N^4(2X_0^2)^\wedge \times [0, 1))$ . But then there is also a more intrinsic description. Now, there is no limit for the growth height. But on some pieces the growth process stops at some finite height; the growth there stops at  $X^3(\alpha) \times \{1\} - \bar{\varepsilon}(X^3(\alpha))$ . Then, on the other pieces, the growth continues indefinitely, up to an infinite height, and this defines the

$$\bar{\varepsilon}(X^3(\alpha)) \subset X^3(\alpha) \times \{1\}.$$

Finally, starting with the diffeomorphism (40.1), which we re-write here:

$$\left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_k \right)^\wedge + \sum_1^{\bar{n}} D^2(\Gamma_j) \stackrel{\text{DIFF}}{=} N^4(\Delta^2),$$

one proves the more serious (51) just like one proves (43), working now with the full  $\sum_\alpha X^3(\alpha)$ . An impressionistic view of these things is presented in the Figure 16. And remember here that  $(\text{LAVA}) \cap \sum \Gamma_j = \emptyset$ .



**Figure 16.**

An impressionistic view of the proof of (58). The step  $(A) \Rightarrow (B)$  is a diffeomorphism, where a whole orchestra of  $X^2(\alpha)$ 's grows simultaneously, in a smoothly controlled manner.

We will move now to the following lighter notation

$$(L_\alpha, \delta L_\alpha) = (\text{LAVA}_\alpha, \delta \text{LAVA}_\alpha)$$

with which, our previous formulae become

$$(L_\alpha, \delta L_\alpha) \stackrel{\text{DIFF}}{=} (\delta L_\alpha \times [0, 1] - \bar{\varepsilon} \delta L_\alpha \times \{1\}, \delta L_\alpha \times \{0\}),$$

$$L_\alpha^\wedge = (\delta L_\alpha \times [0, 1]) \underbrace{\cup}_{\delta L_\alpha} (\delta L_\alpha)^\wedge = (\delta L_\alpha \times [0, 1]) \cup \lambda_\alpha.$$

**A complement to the Lemmas 8 and 9; more on the product property of LAVA.**

1) *There is a PROPER Whitehead collapse*

$$(62) \quad L_\alpha^\wedge \supset \delta L_\alpha \times [0, 1] \xrightarrow{\hat{\pi}_\alpha} \delta L_\alpha.$$

[COMMENT. It is the PROPER collapse, gotten from the (62) above

$$\left( N^4(2\Gamma(\alpha)) - \sum h_n \right) \cup \sum_\alpha \delta L_\alpha \times [0, 1] \xrightarrow{\sum_\alpha \hat{\pi}_\alpha} \left( N^4(2\Gamma(\alpha)) - \sum h_n \right),$$

where  $(N^4 - \sum h_n) \cup \sum_\alpha \delta L_\alpha \times [0, 1] \subset ((N^4 - \sum h_n) \cup \text{LAVA})^\wedge$ , which is behind (50) and (51). This collapse is supposed to proceed **smoothly**, with smaller and smaller amplitude as we approach the  $\varepsilon(N^4(2\Gamma(\infty)) - \sum_1^\infty h_n)$ , so as to yield a **diffeomorphism**

$$\left[ \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_n \right) \cup \text{LAVA} \right]^\wedge \stackrel{\text{DIFF}}{=} \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_n \right)^\wedge.$$

The  $\sum_\alpha \hat{\pi}_\alpha$  above is the inverse of the process via which the  $\sum_\alpha \text{LAVA}_\alpha$  grows naturally and very slowly out of  $\sum_\alpha J \times X^3(\alpha)$ . Also, we will denote by  $\pi_\alpha$  the restriction of  $\hat{\pi}_\alpha$  to  $L_\alpha \subset L_\alpha^\wedge$ .

2) The (62) will be called the **strong** product property of LAVA, so as to distinguished from the **weak** product properties to be developed next, and which is consequence of the strong property. Let us start with a compact bounded, not necessarily connected surface  $(S, \partial S)$  coming with an inclusion

$$(63) \quad (S, \partial S) \xhookrightarrow{i} (\delta L_\alpha, \partial \delta L_\alpha).$$

In principle, at least, we will always think of  $(S, \partial S)$  as consisting of connected components, each a copy of  $(D^2, \partial D^2 = S^1)$ .

This induces embeddings

$$\begin{array}{ccc} (\hat{\pi}_\alpha^{-1} S, \partial S \times [0, 1]) & \longrightarrow & (\hat{L}_\alpha, (\partial \delta L_\alpha) \times [0, 1]) \\ \uparrow & & \uparrow \\ (\pi_\alpha^{-1} S, \partial S \times [0, 1]) & \longrightarrow & (L_\alpha, (\partial \delta L_\alpha) \times [0, 1]), \end{array}$$

with  $\hat{\pi}_\alpha^{-1} S = S \times [0, 1]$ , with  $\hat{\pi}_\alpha^{-1} S \cap \bar{\varepsilon} L_\alpha$  a **tame** totally discontinuous (Cantor) subset of  $S \times \{1\} \subset \hat{\pi}_\alpha^{-1} S$  and with  $\pi_\alpha^{-1} S = \hat{\pi}_\alpha^{-1} S - \hat{\pi}_\alpha^{-1} S \cap \bar{\varepsilon} L_\alpha$ .

3) In this same vein, let

$$x \in \Gamma(2\infty) \cup \sum_1^\infty D^2(C_i)$$

be like in (28.4). Then, in terms of the things just said

$$(64) \quad N^4(\text{extended cocore of } (x)) = \sum_\alpha \pi_\alpha^{-1}(x)$$

and this is a copy of  $B^3 = \{\text{a } \mathbf{tame} \text{ Cantor set} \subset \partial B^3\}$ . From now on, we simplify the notation and, unless the opposite is explicitly said, the (64) above will be denoted just by

$$(65) \quad \{\text{Extended cocore}(x)\} \subset \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_n \right) \cup \text{LAVA},$$

and notice here the capital “E”. The (65) is a *PROPER* codimension one submanifold, which can be compactified into

$$\{\text{Extended cocore}(x)\}^\wedge \equiv \sum_\alpha \hat{\pi}_\alpha^{-1}(x) \subset \left[ \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_n \right) \cup \text{LAVA} \right]^\wedge.$$

There is a diffeomorphism  $\{\text{Extended cocore}(x)\}^\wedge \stackrel{\text{DIFF}}{=} B^3$ .

4) We have a diffeomorphism of pairs

$$(66) \quad \left( \left[ \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_n \right) \cup \text{LAVA} \right]^\wedge, \sum_1^n \{\text{Extended cocore}(R_i)\}^\wedge \right) \\ \stackrel{\text{DIFF}}{=} n \# ((S^1 \times B^3), (* \times B^3)) \text{ (standard)}.$$

Here, the  $\sum_1^n R_i \subset \Gamma(1)$  correspond to the RED 1-handles of  $N^4(\Delta^2)$ .

In what follows next, in the present section we will make more explicit the structure of LAVA. The next items will not be used explicitly, later on, in the proof of our main result, but they may clarify things.

We will call **elementary** a matrix  $\mathcal{M}_0$ , à la (30), which is both of the easy id + nilpotent type and which has a unique minimal (i.e. final) state  $i_1 = h_{i_1} \cup B_{i_1}$ . We will assume  $\mathcal{M}_0$  to be a **full submatrix** of our  $C \cdot h = \text{id} + \text{nilpotent}$ , meaning that no state in  $\mathcal{M}_0$  receives arrows from outside  $\mathcal{M}_0$ , when we consider  $\mathcal{M}_0$  as a piece of  $C \cdot h$ . There is a contribution of  $\mathcal{M}_0$  to LAVA, a LAVA( $\mathcal{M}_0$ ) or LAVA $^\wedge$ ( $\mathcal{M}_0$ ) and we will refer to these as being an **atom**, respectively a **compact atom**. We will use the following generic notation for the successive states of  $\mathcal{M}_0$

$$i_1(\text{unique minimal state}) \leftarrow i_2(+i_2' + i_2'' + \dots) \leftarrow i_3(+\dots) \leftarrow \dots$$

Let now  $S \subset \partial(i_1)$  be a SPOT looking like one of the  $\text{Im}(\nu(i \rightarrow j_k))$  in Figure 12. Of course our present  $S$  cannot be a  $\text{Im}(\nu(i_1 \rightarrow \dots))$ , since the state  $i_1$  is minimal.

At the level of  $2X_0^2$  the  $\pi(\mathcal{M}_0)^{-1}S$  is a full tree, splitting locally  $2X_0^2$ , except at its foot  $S$ . When one goes  $4^d$ , then at the level of  $N^4(2X_0^2)$  our  $\pi(\mathcal{M}_0)^{-1}S$  becomes an object which I will denote by  $M^3[S] \stackrel{\text{DIFF}}{=} B^3 - \{\text{a tame Cantor set} \subset \partial B^3\}$ , organized as follows

$$(67) \quad M^3[S] = X_1 \# X_2 \# X_3 \# \dots,$$

where  $X_1 = \{\text{the contribution } B^3(i_1) \text{ of } i_1\}$ ,  $X_2 = \{\text{the contribution of } i_2\}$ , which is a finite collection of  $B^3(i_2)$ 's, with  $\#$  a multiple connected sum, a.s.o. Let us say that  $X_n = \sum_{j=1}^{N(n)} B_j^3(i_n)$ . The Figure 17 suggests the structure of

$$([M^3(S)]^\wedge, [M^3(S)]) = (\hat{\pi}(\mathcal{M}_0)^{-1}(S), \pi(\mathcal{M}_0)^{-1}(S))$$

coming with

$$F_S = [M^3(S)]^\wedge - [M^3(S)],$$

which is a Cantor set suggested in Figure 17, by fat points.

Because of typographical reasons, in the Figure 17 we did not represent realistically the organic structure of  $[M^3(S)]^\wedge$ , which we draw as a square.

For the triplet

$$(\text{compact atom}; \delta(\text{atom}), \text{atomic lamination}),$$

where  $\delta(\text{atom})$  splits the compact atom from the rest of the universe, we have the explicit description below

$$(68) \quad ([M^3(S)]^\wedge \times [0, 1]; ([M^3(S)] \times \{0\}) \cup (S \times [0, 1]) \cup ([M^3(S)] \times \{1\}), \\ F_S \times [0, 1]) = (\text{LAVA}(\mathcal{M}_0)^\wedge; \delta \text{LAVA}(\mathcal{M}_0), \bar{\varepsilon}(\mathcal{M}_0)).$$

We move now to a completely general geometric intersection matrix (30) which is of the easy id + nilpotent type, without being necessarily elementary. We will show now how the  $\text{LAVA}_\alpha$  and, more precisely the triple

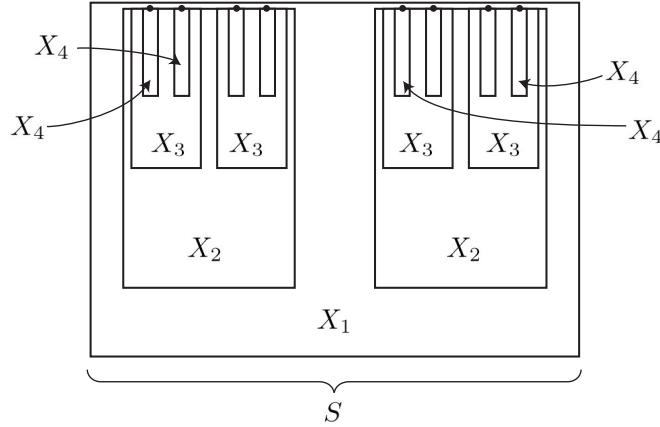
$$(\text{LAVA}_\alpha^\wedge; \delta \text{LAVA}_\alpha, \varepsilon(X^3(\alpha)))$$

can be reconstructed by glueing together in a locally finite but not necessarily simply-connected manners, compact atoms of type (68).

We start by introducing, on the same lines as  $X^3(\alpha)$  in (35), the following object

$$(69) \quad X^1(\alpha) = \left( \sum_{i \in \alpha} I_i \right) / \mathcal{R}^1 (= \{\text{the equivalence relation induced by the various } \lambda(i \rightarrow j)\}).$$

Topologically speaking, this is an infinite, connected, locally finite, highly non-simply-connected graph. But its natural structure is not the standard edge/vertex structure; instead we have a richer kind of structure which we will describe. Roughly speaking  $X^1(\alpha)$  is a **train-track** with infinite weights, recording the way in which our various  $I_i$ 's go through each other. Each  $I_i$  is here a **smooth** arc and the maps  $\lambda(i \rightarrow j)$  are smooth too.



**Figure 17.**

With one dimension less, this figure should suggest the 3-ball

$$[M^3(S)]^\wedge = X_1 \# X_2 \# \dots \# X_n \# \dots \cup \{\text{Cantor set } F_S\},$$

with  $S \subset \partial X_i$ . LEGEND:  $\bullet$  = points in  $F_S$ .



We have obvious maps

$$(70) \quad X^3(\alpha) \xrightarrow[\text{"quotient-space projection"}]{P} X^1(\alpha) \xleftarrow[\text{inclusion of a DISCRETE subset}]{} X^0(\alpha) \stackrel{\text{def}}{=} \sum_i \partial I_i.$$

All the **branching points** of  $X^1(\alpha)$  i.e. the points where, locally speaking  $X^1(\alpha)$  fails to be a smooth line, occur at the  $X^0(\alpha)$  points. For any  $p \in X^1(\alpha) - X^0(\alpha)$  we can define an elementary  $\mathcal{M}_0(p)$ , which should be thought of as the **weight** of  $p$ , as follows.

(71) We will start by introducing  $J(p) \equiv \{\text{the set of } I_i\text{'s which are such that } p \in I_i, \text{ counted with multiplicities; any given } I_i \text{ may go finitely many times through the point } p\}$ .

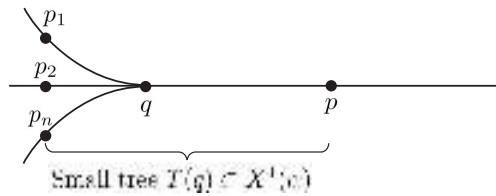
By definition, this will be the set of states of  $\mathcal{M}_0(p)$ . Then, for  $I_i, I_j \in J(p)$ , any occurrence  $p \in I_j \subsetneq I_i$ , without intermediary  $p \in I_j \subsetneq I_k \subsetneq I_i$ , will be, by definition, an arrow  $\{i \rightarrow j\} \in \mathcal{M}_0(p)$ . End of (71).

**Remarks.** a) Notice that the  $I_i$ 's, states of  $J(p)$ , are occurrences  $p \in I_i$ , not just  $I_i$ 's. Consider then two distinct occurrences  $p \in I_j \subset \overset{\circ}{I}_i$ ,  $p \in I_k \subset \overset{\circ}{I}_i$ . We may have here  $j = k$  or  $j \neq k$ . If  $j = k$ , then these occurrences  $p \in I_j$ ,  $p \in I_j$  are distinct states of  $J(p)$  (hence so are the  $p \in I_i$ ,  $p \in I_i$ ) and we have two disjoint edges in  $\mathcal{M}_0(p)$ . If  $j \neq k$ , one can show that we have two successive arrows in  $\mathcal{M}_0(p)$ . It follows from these things that **a state in  $J(p)$  has at most one outgoing arrow**.

b) To each smooth point  $p \in X^1(\alpha)$  as above, corresponds a properly embedded disk  $S(p) \subset X^3(\alpha) = \delta \text{LAVA}_\alpha$  and, if we consider

$$\text{LAVA}_\alpha \xrightarrow{\pi_\alpha} \delta \text{LAVA}_\alpha = X^3(\alpha) \xrightarrow{P} X^1(\alpha),$$

then  $\pi_\alpha^{-1}S(p) = \{\text{the } \pi(\mathcal{M}_0)^{-1}S(p) \text{ from (67)}\} \equiv [M^3(p)]$ . This fiber stays constant along connected components of  $X^1(\alpha) - X^0(\alpha)$  but it **jumps** at points in  $X^0(\alpha)$ . Roughly speaking, **this is the correct description of  $\text{LAVA}_\alpha$** ; it will be rendered more precise, below. The typical structure of the train-track  $X^1(\alpha)$  in the neighbourhood of a point  $q \in X^0(\alpha)$  is suggested in the Figure 18.



**Figure 18.**

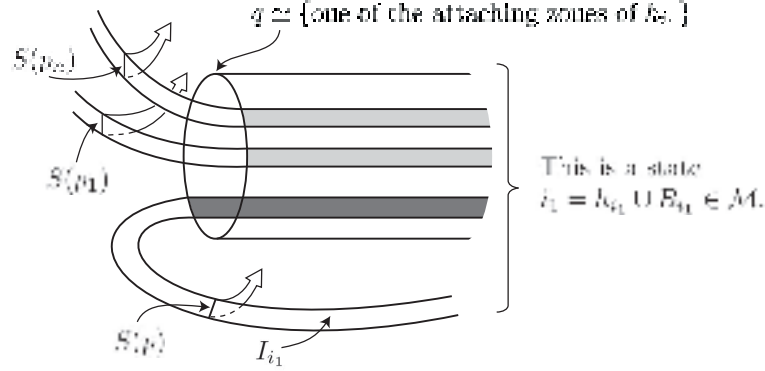
Here  $n = n(q) < \infty$ . We see a small piece of  $X^1(\alpha)$ ,  $q \in X^0(\alpha)$  and  $p, p_i$ 's are smooth points.

Together with each  $q \in X^0(\alpha)$  (see Figure 18) comes a smooth PROPER embedding

$$\sum_1^n [M^3(p_i)] \xrightarrow{\xi(q)} [M^3(p)]$$

which is such that  $\text{Im } \xi(q) = \{\{X_2 \cup X_3 \cup \dots\} \text{ of } p, \text{ (see (67))}\}$ , while  $\xi(q)(X_j(p_i)) \subset X_{j+1}(p)$ , for each  $1 \leq i \leq n(q)$ .

In terms of the Figure 12, which in real life is of course supposed to be 4-dimensional, the purely 1<sup>d</sup> Figure 18 corresponds to the Figure 19.



**Figure 19.**

This is what Figure 18 (which is in  $X^1(\alpha)$ ) corresponds to, at the level of  $X^3(\alpha)$  and/or  $\text{LAVA}_\alpha$ . Here  $p, p_1, \dots, p_n$  are smooth points of  $X^1(\alpha)$  (i.e. points of  $X^1(\alpha) - X^0(\alpha)$ ). At the level of  $X^1(\alpha)$  there is a unique occurrence  $p \in I_{i_1}$  and this is the unique minimal state in  $J(p) \subset \mathcal{M}_0(p)$ . We also have

$$[M^3(p)] = \{\text{extended cocore } S(p)\}.$$

Notice that here  $S(p) + S(p_1) + \dots + S(p_n)$  are properly embedded inside  $\delta \text{LAVA}_\alpha$ . Apart from small scallops coming from the shaded areas, the attaching zone  $\partial J_{i_1} \times B_{i_1}^*$  (see (31)) of the 1-handle  $h_{i_1}$  is also in  $\delta \text{LAVA}_\alpha$ . So, the discs  $S(p) + S(p_0) + \dots + S(p_n)$  split out of  $\delta \text{LAVA}_\alpha$  a 3-ball  $B^3(q)$ .

Here is the description of the  $(P \circ \pi_\alpha)^{-1} T(q) \subset \text{LAVA}_\alpha$ , for a neighbourhood  $T(q)$  of  $q \in T(q) \subset X^1(\alpha)$ , like in the Figure 18

$$(72) \quad (P \circ \pi_\alpha)^{-1}(T(q)) = \sum_{i=1}^{n(q)} [M^3(p_i)] \times [1, 0] \cup [M^3(p)] \times [0, 1],$$

where the two pieces are glued together along

$$\sum_i [M^3(p_i)] \times \{0\} \approx \text{Im } \zeta(q) \subset [M^3(p)] \times \{0\}$$

and where

$$\sum_i [M^3(p_i)] \times \{1\} + [M^3(p)] \times \{1\}$$

splits the  $(P \circ \pi_\alpha)^{-1} T(q)$  from the rest of the  $\text{LAVA}_\alpha$ .

Let  $m_1, m_2, \dots$  be the minimal states of  $\mathcal{M}$ . For each  $I_{m_i}$  we consider some  $I'_{m_i} \subset \text{int } I_{m_i}$ . For the generic  $p \in \text{int } I_{m_i}$  we have then

$$(73) \quad (P \circ \pi_\alpha)^{-1} I'_{m_i} = [M^3(p)] \times I'_{m_i}, \text{ like in (68).}$$

One can find a family of smooth points  $\mathcal{P} \subset X^1(\alpha)$ , s.t.

$$\{X^1(\alpha) \text{ SPLIT along } \mathcal{P}\} = \sum_{q \in X^0(\alpha)} T(q) + \sum_{m_i} I'_{m_i},$$

yielding

(74)

$$\begin{aligned} \text{LAVA}_\alpha^\vee &\equiv \{ \text{LAVA}_\alpha \text{ SPLIT along } \sum_{p \in \mathcal{P}} \{ \text{extended cocore } (p) \} \} \\ &= \sum_{q \in X^1(\alpha)} \{ (P \circ \pi_\alpha)^{-1}(T(q)) \} + \sum_{m_i} \{ (P \circ \pi_\alpha)^{-1} I_{m_i} \}. \end{aligned}$$

Next, glueing back together, in the obvious way, the spare pieces in (74) we finally get our  $\text{LAVA}_\alpha$ . One can also extract from here a more or less explicit of

$$\mathcal{L}_\alpha \subset \text{LAVA}_\alpha^\wedge \supset \text{LAVA}_\alpha \supset \delta \text{LAVA}_\alpha.$$

## 4 Constructing exterior discs, which are disjointly embedded

We return now to  $X^3(\text{old}) = \{3\text{-skeleton of int } \Delta_1^4\}$  and to its RED  $3^d$  collapse  $X^3(\text{old}) \searrow \Delta^3 \equiv \{3\text{-spine of } \Delta_{\text{Schoenflies}}^4\}$ . This collapse kills teh 2-cells  $D^2(\gamma_k^0) \subset X^2(\text{old}) - \Delta^2 (= 2 \text{ skeleton of } \Delta^4)$  (with  $\Delta^2 \subset \Delta^3$ ). More explicitly for every  $\gamma_k^0 \subset \{\text{link}\}$  (9) we have a simplicial nondegenerate map, with a collapsible source,

$$(75) \quad B^3(\gamma_k^0) \xrightarrow{F_k} \overline{X^3 - \Delta^3},$$

with the following features. We have  $D^2(\gamma_k^0) \subset \partial B^3(\gamma_k^0)$  and we introduce the 2-cell, coming with its non-degenerate map

$$d_k^2 \equiv \partial B^3(\gamma_k^0) - \text{int } D^2(\gamma_k^0) \xrightarrow{f_k \equiv F_k|_{d_k^2}} X^2(\text{old}).$$

With this, our  $F_k$  (75) is injective except for possible edge-effects, meaning double points of the  $f_k$  above.

Notice that  $\gamma_k^0 = \partial d_k^2$ . Also, the  $F_k(B^3(\gamma_k^0))$ 's are disjointed, except for the following two items:

- ) edge effects,
- ) telescopic embeddings  $F_p(B^3(\gamma_p^0)) \subset F_k(B^3(\gamma_k^0))$  induced by  $\gamma_p^0 \subset d_k^2 - \gamma_k^0$ .

All this, so far, was in the context of  $X^2(\text{old})$  and we move now to  $X^2[\text{new}]$  forgetting about the  $3^d$  collapse. Every time for our  $d_k^2$  above we find  $D^2(\Gamma_j) \times (\xi_0 = 0) \subset f_k d_k^2$ , we replace it by  $\Gamma_j \times [0 \geq \xi_0 \geq -1] \cup (D^2(\Gamma_j) \times (\xi_0 = -1))$  getting this way a **new**  $(d_k^2, f_k)$ , the only one to be used from now on, and we will not bother to change the notations

$$(76) \quad \begin{array}{ccc} d_k^2 & \xrightarrow{f_k} & 2X_0^2 \\ & \searrow & \nearrow \\ & X_0^2[\text{new}] = X_0^2 \times r. & \end{array}$$

One should notice that when we go to the context  $X_0^2[\text{new}]$ , then to the  $\gamma_k^0 \subset X^2(\text{old})$ , we have to add the curves  $\Gamma_j \times (\xi_0 = 0)$ ,  $\Gamma_j \subset \Delta^2$ .

We will work, in what follows from now in this paper, in the following context which supersedes the (2), for the time being

(77)

$$\begin{array}{ccccccc} N^4(\Delta^2) & \subset & N^4(X_0^2 \times r) & \subset & N^4(2X_0^2)^\wedge & \subset & N_1^4(2X_0^2)^\wedge \equiv N^4(2X_0^2)^\wedge \underbrace{\cup}_{\partial N_1^4(2X_0^2)^\wedge} [\partial N^4(2X_0^2)^\wedge \times [0, 1]]. \\ \uparrow & & \uparrow & & \uparrow & & \\ N^4(\Gamma(1)) & \subset & N^4(\Gamma_1(\infty)) & \subset & N^4(2\Gamma(\infty)) & & \end{array}$$

We have

$$N^4(X_0^2 \times r) = N^4(\Gamma_1(\infty)) + \sum_{C, \Gamma} \{\text{the } 4^{\text{d}} \text{ 2-handle } D^2(C, \Gamma)\},$$

where  $(C, \Gamma) \in \{\text{the } \{\text{link}\} [\text{new}] \text{ from (28.2), with all the } \gamma^0\text{'s DELETED}\}$ . The  $N^4(X_0^2 \times r)$  is the place where a large part of the action of this paper takes place and, when we write  $N^4(\Gamma_1(\infty))$  rather than just  $N^4(\Gamma(\infty))$ , it is because we want the  $\Sigma b^3(\beta) \subset \partial N^4(\Gamma_1(\infty))$  to be with us.

Let now the set  $B[\text{new}]$  be like in the context of  $X^2[\text{new}]$ , meaning  $B[\text{new}] \equiv B(\text{old}) + \{\text{For every edge } e_i \subset \Gamma(1) \subset \Delta^2, \text{ there is now a } b_i \in e_i \times (\xi_0 = -1), \text{ the dual curve of which is the}$

$$\eta_i = \partial[e_i \times [0 \geq \xi_0 \geq -1]]\}.$$

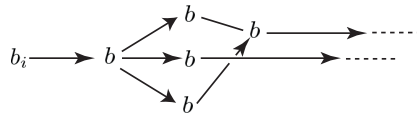
At the level of  $X^2[\text{new}]$ , the  $B[\text{new}]$  has an order relation given by the geometric intersection matrix  $\eta \cdot B = \text{id} + \text{nil}$  of the easy type, of  $X^2[\text{new}]$ . This natural BLUE order induced by  $\eta \cdot B$  is defined as follows. The off-diagonal terms  $\eta_\ell \cdot B_k$  ( $\ell \neq k$ ) define an oriented graph of vertices  $B[\text{new}]$  and oriented edges  $\ell \rightarrow k$ . The  $\text{id} + \text{nil}$  condition implies that there are no closed oriented orbits of the graph, and the oriented edge  $\ell \rightarrow k$  will mean  $\ell > q$ , generating a partial order relation.

(77.1) In the ordered set  $B[\text{new}]$ , pertaining to the space  $X^2[\text{new}] \supset X_0^2[\text{new}] \approx X_0^2 \times r \subset 2X_0^2$ , with  $X^2[\text{new}] \subset 2X^2$ , the elements

$$(b_i, \eta_i) = (b_i \times (\xi_0 = -1), \partial(e(b_i) \times [0 \geq \xi_0 \geq -1])),$$

are INITIAL elements, without incoming arrows (in the ordered graph) i.e. without higher indices  $j > i$  is the ordered set  $B[\text{new}]$ . **[Remark.** Contrary to what happened with the  $B$  of  $X^2$  ( $= X^2(\text{old})$ ), in the context of Lemma 4 and the “Important Digression” which follows it,  $B[\text{new}]$  is not totally ordered, we only have a partial order relation now. No question now any longer, of indexing the states by  $Z_+$  or by  $Z$ .] End of (77.1).

Very importantly, it is the BLUE  $2^{\text{d}}$  collapse of  $X^2[\text{new}]$  which concerns us now and that is what (77.1) talks about, and NOT the one of  $2X^2$ . In the BLUE  $2^{\text{d}}$  collapse which interests us now, every  $b_i \in \Gamma(1) \times (\xi_0 = -1)$  comes, inside the ordered graph corresponding to  $\eta \cdot B = \text{id} + \text{nil}$ , with a trajectory which, a priori, could have been something like below



stopping, in finite time at some dead ENDS.

Actually, the GPS structure which we are using, the condition (5)-(d) and the strategy developed in Figure 35.2 and described afterwards, imposes for  $b_i = b_i \times (\xi_0 = -1) \in \Delta^2 \times (\xi_0 = -1)$  a much more restricted form than the one above. There are two cases:

I) The trivial case when  $b_i \times (\xi_0 = 0) \notin B_0$ . Then  $b_i = b_i \times (\xi_0 = -1)$  itself is a DEAD END. Theorem 11 for the corresponding  $\delta_i^2$  is now trivially verified and we will not dwell longer on this case.

II) The  $b_i \times (\xi_0 = 0) \subset X^3 \times t_j$  is in  $B_0$ . Then, to begin with, inside  $X^3 \times t_j$ , we have a main **linear** trajectory of  $b_i \times (\xi_0 = -1)$ , call it  $\{\text{reduced } T(b_i)\}$

$$(77.1.\text{bis}) \quad b_i = b_i \times (\xi_0 = -1) \rightarrow \underbrace{b_i \times (\xi_0 = 0) = b_{i_1} \rightarrow b_{i_2} \rightarrow b_{i_3} \rightarrow \dots \rightarrow b_{i_k}}_{\text{inside } X^3 \times t_j} \text{ (DEAD END)}.$$

So, in its full glory, we have a tree  $T(b_i) \supset \{\text{reduced } T(b_i)\}$ , representing the complete BLUE trajectory of  $b_i \times (\xi_0 = -1)$

some of these may either not exist or be doubles. But anyway, they are all DEAD ENDS.

[The “double” here comes from the fact that  $D^2(\eta(b_{i..}))$  has two vertical edges.]

Let  $b_j$  be the generic  $b \in \bigcup_{b_i \in (\xi_0 = -1)} T(b_i)$ . To this  $b_j$ , in  $X^2[\text{new}]$  corresponds a  $D^2(\eta_j) \subset X^2[\text{new}]$ .

[But then, remember, that at the level of  $2X^2 \supset 2X_0^2$  this  $D^2(\eta_j)$  is now a  $D^2(\gamma^1)$  (in the extended sense).]

For our 2-cell  $D^2(\eta_j) \subset X^2[\text{new}]$ , the BLUE curve  $\eta_i$  has a RED counterpart  $\Gamma_\ell$  or  $C_i$  or  $\gamma_k^0$  in the link (9) and to this corresponds a  $[D^2(\eta_j)] \subset X^2[\text{new}]$  defined as follows: If  $\eta_j = \Gamma_\ell$  or  $C_i$ , then  $[D^2(\eta_j)] = \{D^2(\Gamma_\ell)$  or  $D^2(C_i)\}$ , i.e. in this case  $[D^2(\eta_j)] = D^2(\eta_j)$ , topologically speaking, as pieces of  $X_0^2[\text{new}]$ . If  $\eta_j = \gamma_k^0$ , then  $[D^2(\gamma_j)]$  stands just for the collar  $\gamma_k^0 \times [0, 1]$ , where  $\gamma_k^0 \times \{1\} \equiv \gamma_k^0 \subset \partial X_0^2[\text{new}]$ , and, of course, there is no  $\{\text{full } D^2(\gamma^0)\} \subset X_0^2[\text{new}]$ . Here  $b_j \in e(b_j)$  is like in Figure 7-(II, III), when we are in the generic case, far from the root of the trees  $T(b_i)$ , respectively like in Figure 7-bis for  $b_i \in \Gamma(1) \times (\xi_0 = -1)$ , in the non-generic case. To the  $b_j$  we attach the following sphere with holes  $B_j^2 \subset 2X_0^2$ , defined as follows:

(77.2) When we are in the generic case, then

$$B_j^2 \equiv \{([D^2(\eta_j)] \times r) \cup (\eta_j \times [r, b]) \cup (D^2(\eta_j) \times b, \text{ which is always there})\},$$

and here  $e(b_j) \times [r, b]$  is replaced by the surviving collar in the Figure 7-(II, III), and the same for every  $e(b_k) \times [r, b]$ , where  $b_k \subset \eta_j - b_j$ , coming clearly with an arrow  $b_j \rightarrow b_k$  in the oriented graph defined by the  $\eta \cdot B$  of  $X^2_{\text{new}}$ .

In the case when we consider a  $b_i \in \Gamma(1) \times (\xi_0 = -1)$ , then  $B_i^2$  is defined like above, EXCEPT that now  $e(b_i) \times [r, b]$  is like in Figure 7-bis, with all of its interior removed. Of course  $e(b_i) \subset \eta_i$ .

Since the special treatment of Figure 7-bis is restricted to  $e(b_i)$  itself, its effect stays confined at  $\xi_0 = -1$  and, with this, in **all** cases, the  $B_i^2$  is a sphere with holes (see here Figure 19.1 too). This is coming with

(78)  $\partial B_j^2 = c(b_j) + \{\text{some } c(b_{\ell < j}) \text{ getting arrows } b_j \rightarrow b_\ell, \text{ see here the order considered in (77.1)}\} + \{\text{possibly a } \gamma_k^0\text{'s, when } \eta_j = \gamma_k^0 \text{ (RED-wise, in } X_0^2[\text{new}])\} \subset \partial(2X_0^2)$ . [The  $\gamma_k^0$  here could be  $\Gamma_j \times (\xi_0 = 0)$ ].

[The BLUE order in  $X^2[\text{new}]$  is being meant here, organized like in the IMPORTANT DIGRESSION which follows after Lemma 4, adapted afterwards from  $X^2$  to  $X^2[\text{new}]$ . Also, in the generic case, the  $c(b_j)$  means the smaller of the two boundary curves, of the shaded collar in the case of Figure 7-(II, III), while in the case  $b_i \times (\xi_0 = -1) = b_i$ ,  $c(b_i)$  is the boundary of the rectangle in Figure 7-bis. Notice that  $X^2[\text{new}]$  is used for the BLUE order, while  $2X_0^2$  is used for providing the spare parts for the  $B_i^2$ 's.]

We will denote from now on by  $\sum_1^M b_i$  the family of the  $b_i \in \Delta^2 \times (\xi_0 = -1)$ , which is occurring in (77.1) to (77.1-ter).

**Lemma 10.** 1) For each  $b_i \in \sum_1^M b_i$  there is a disc  $\delta_i^2$ , cobounding  $c(b_i)$  (i.e.  $\partial\delta_i^2 = c(b_i)$ ), endowed with a non-degenerate map

$$(79) \quad \delta_i^2 \xrightarrow{g_i} 2X_0^2,$$

and this  $\delta_i^2$  is gotten by the following inductive construction, the induction using here the **BLUE order** of  $X^2[\text{new}]$ . The construction will only make use of the outgoing trajectories of  $\sum_1^M b_i$ , see here the (77.1-ter).

(79-bis) Here is how we **CONSTRUCT THE DISC**  $\delta_i^2$ . We start with the disjointed union of all the  $B_j^2$ 's (see (77.2)) for all the  $b_j^2 \in \{\text{reduced } T(b_i)\}$  (77.1-bis). To these, we add now the additional  $b(\alpha)$ 's from the **full**  $T(b_i)$  (77.1-ter).

These  $b(\alpha)$ 's, seable in Figure 35.3-(B) appear on the boundary of a vertical 2-cell  $e(b(\alpha)) \times [t_j, t_{j\pm 1}]$  and they also come with an annulus  $c(b(\alpha)) \times [0, \varepsilon]$  in the Figure 7-(II, III). Here  $c(b(\alpha)) \equiv \{\text{outer border of the annulus}\} = \{[e(b(\alpha)) \times r (= \text{our } e(b(\alpha)))] \cup \partial e(b(\alpha)) \times [r, b] \cup [e(b(\alpha)) \times b]\}$ , lives at  $t_j$  and bounds a thin  $[r, b]$ -rectangle, call it  $R^2(b(\alpha))$ , normally living at  $t = t_j$  too.

In  $2X_0^2$ ,  $c(b(\alpha)) = c(b(\alpha)) \times t_j$  bounds the disc

$$(79.0) \quad B^2(b(\alpha)) \equiv (c(b(\alpha)) \times [t_j, t_{j\pm 1}]) \cup R^2(b(\alpha)) \times t_{j\pm 1},$$

to which we may or may not add the annulus  $c(b(\alpha)) \times [0, \varepsilon]$ , shaded in Figure 7-(II, III), according to our convenience, see what is said after the (79.1) below.

In order to get the  $\delta_i^2$  we start by putting together the  $B_j^2$ 's from (77.2) along the common boundary curves  $c(b_j)$ . [To be more precise, when we have  $b_{j_1} \rightarrow b_{j_2}$  in the oriented graph  $\mathcal{M}(\eta \cdot B$  of  $X^2[\text{NEW}]$ ) then, to begin with,  $c(b_{j_2}) \times [0, \varepsilon] = \{a \text{ thin boundary collar, of exterior boundary } c(b_{j_2}) \times \{\varepsilon\}\}$ , occurs both in  $B_{j_1}$  and in  $B_{j_2}$ . In  $\delta_j^2$  we throw then in  $B_{j_1} \cup B_{j_2} - \{\text{the common } c(b_{j_2}) \times (0, \varepsilon]\}$ , rendered now smooth. In this construction, each  $c(b_{\ell < j}) \subset \partial B_j^2$  (i.e. with  $j \rightarrow \ell$ ), continues inductively with its  $B_\ell^2$ . Moreover, when  $\gamma_k^0 \subset \partial B_j^2$  OR  $c(b(\alpha)) \subset \partial B_j^2$  occur, then we fill in with  $d_k^2$ , the one cobounding the  $\gamma_k^0$ , respectively with  $B^2(b(\alpha))$  (see (79.0)). In the context of (79), when we get to the  $d_k^2$  we set  $g_i \mid d_k^2 = f_k$ . End of (79-bis).

**Important remark.** In order to construct  $\delta_i^2$  we need  $2X_0^2$ , which provides the necessary spare parts. But the order in which we put things together, is the BLUE order of  $X^2[\text{NEW}]$ , **not** the one of  $2X^2$ .  $\square$

Given the form of  $T(b_i)$  (77.1-ter) our  $\delta_i^2$  is a disc of boundary  $c(b_i)$

$$(79.1) \quad \delta_i^2 = \{a \text{ main disc with holes } \mathcal{B}_i^2 \equiv \bigcup \{\text{the contribution of the } B_j^2 \text{'s and } B(b(\infty)) \text{'s}\} \cup \sum d_k^2 \text{'s, filling in the holes.}$$

[The  $\mathcal{B}_i^2$  normally contains the contribution  $B^2(b(\alpha))$  (see (79.0)) for each  $b(\alpha)$  involved in the full  $T(b_i)$ . But, for purely expository purposes, we will chose to let only the collar  $c(b(\alpha)) \times [0, \varepsilon]$  occur in  $\mathcal{B}_i^2$ , to begin with, throwing in the new boundary  $c(b(\alpha)) \times \varepsilon$  much later than the boundaries  $\gamma_k^0 \times \varepsilon$  of  $\mathcal{B}_i^2$ . This way we avoid for  $c(b(\alpha))$  the **deletions** of the  $c(b_{j_2}) \times [0, \varepsilon]$ 's mentioned earlier in our construction of  $\delta_i^2$ .]

We will work from now on with the following big but finite map

$$(**) \quad \sum_1^M \delta_i^2 \xrightarrow{\sum_1^M g_i} 2X_0^2.$$

We denote by  $\sum_1^N d_k^2$  the  $d_k^2$ 's occurring in the map above. Here  $N = N(M)$ .

2) The global map (\*\*) above can be lifted off  $2X_0^2$  and changed into a generic immersion (the generic form of maps of  $2^d$  into  $4^d$ ), which is winding tightly around  $2X_0^2 \subset N_1^4(2X_0^2)^\wedge$ ,

$$(80) \quad \sum_1^M \delta_i^2 \xrightarrow{J} N_1^4(2X_0^2)^\wedge;$$

this map  $J$  extends the  $\sum_1^M c(b_i) \subset \partial N^4(2X_0^2)^\wedge \subset N_1^4(2X_0^2)^\wedge$ .

3) Generally speaking, the  $J$  (80) has ACCIDENTS, namely double points  $x \in JM^2(J) \subset N_1^4(2X_0^2)^\wedge$  AND also transversal contacts

$$(81) \quad z \in J\delta_i^2 \pitchfork 2X_0^2 \subset N_1^4(2X_0^2)^\wedge.$$

[**Notations.** For any, generic, map  $K \xrightarrow{\psi} L$ , we use the notations  $M_2(\psi) \equiv \{\text{set of } x \in K \text{ s.t. } \# \psi^{-1}\psi(r) > 1\} \subset K$ . But we can also consider the double points of  $\psi$  in the context

$$K \times K \supset M^2(\psi) \xrightarrow{\text{projection on the first factor}} M_2(\psi) \subset K,$$

coming with  $\psi M^2(\psi) = \psi M_2(\psi) \subset L$ .]

The next item is a consequence of (21.A).

4) When it comes to the accidents (81) of the form

$$z \in J\delta_j^2 \pitchfork \Delta^2 \times (\xi_0 = -1),$$

then these can only come from the parts  $d_k^2 \subset \delta_i^2$  and **never** from the parts  $B_i^2 \subset \delta_j^2$ . Here is an immediate consequence of this fact.

(81.1) Every transversal contact

$$z \in JB_i^2 \cap 2X_0^2 \subset N_1^4(2X_0^2)^\wedge$$

possesses an  $\{\text{extended cocore}(z)\} \subset 2X_0^2$ , since it does not concern  $\Delta^2 \times (\xi_0 = -1)$ .

Before going into the proof, notice that in the context of this construction for  $c(b_\ell) = \partial\delta_\ell^2$  we have  $c(b_\ell) \cdot b_k = \delta_{\ell k}$  so, in some singular sense, the  $\sum_1^M \delta_\ell^2$  are in cancelling position with the 1-handles  $\sum_1^M b_\ell$ . We say here “singular”, because of the fact that there are ACCIDENTS, see above.

**Proof.** We only need to prove 4). We start by reminding that any edge  $e_i \subset \Gamma(1) \times (\xi_0 = -1)$  contains a  $b_i \in B[\text{new}]$ . Remember also, that any  $\sigma^2 \subset \Delta^2 \times (\xi_0 = -1)$  is, BLUE-wise a  $D^2(\gamma^1)$  of  $X^2[\text{new}]$ ; so these  $\sigma^2$ 's do **not contribute** to the 2-cells  $D^2(\eta)$  of (77.1). Then, let  $b_i \subset e \subset \partial\sigma^2 \subset \sigma^2 \subset \Delta^2 \times (\xi_0 = -1)$ . As we know, such a  $b_i$  is automatically an INITIAL element in the BLUE order of  $X^2[\text{NEW}]$ , i.e. it has no incoming BLUE arrows. At the level of our  $X_r^2 = X_0^2[\text{NEW}] \subset X^2[\text{NEW}]$ , the geometric intersection matrix  $\eta \cdot b$  is of the easy id + nil type. The BLUE dual of the  $b_i \subset e_i \subset \Gamma(1) \times (\xi_0 = -1)$  is  $D^2(\eta_i) = e_i \times [-1 \leq \xi_0 \leq 0]$ , with  $c(b_i) = \partial(e_i \times [r, b])$  and, if the edge  $e_i \times (\xi_0 = 0)$  contains a  $b_j \in B[\text{NEW}]$ , this comes with  $j < i$ . In this case, the construction of  $\delta^2(b_i)$  does not stop short at  $B_i^2$ , but it continues further.

All the  $B_\ell^2$ 's present in our construction come from the finite family of those  $b \in \sum_1^M T(b_i)$  (see (77.1-bis)).

If, by any chance, for some  $b_i = b_i \times (\xi_0 = -1) \subset \Delta^2 \times (\xi_0 = -1)$  it so happens the  $b_i \times (\xi_0 = 0)$ , which should be next to our  $b_i$  in  $T(b_i)$ , is NOT in  $B[\text{new}]$ , then our  $\delta_i^2$  is trivially the  $B_i^2 = D^2$  and there are no accidents for it. This really is a very trivial case when everything which we will want to achieve is already automatically there. We will ignore this case, from now on.

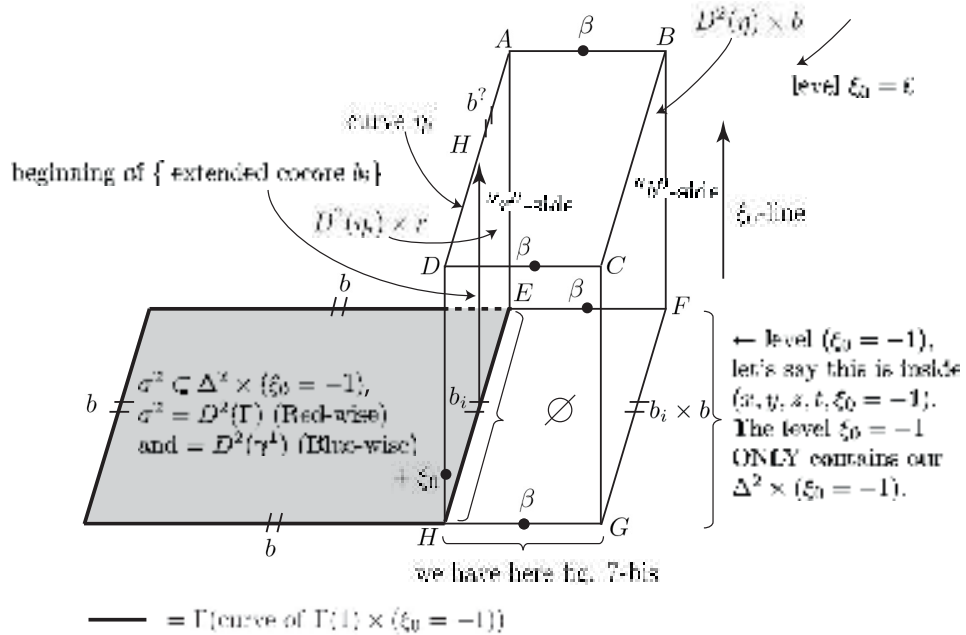
The edges  $P \times [0 \geq \xi_0 \geq -1]$  never contain  $B$ 's and so, for our  $b_j$  above, occurring as  $b^?$  in Figure 19.1, this is the only prospective  $b_{j < i} \subset \eta_i$ , for the  $b_i \in \Gamma(1) \times (\xi_0 = -1)$ .

The edge  $e = e(b_i) \subset \Gamma(1) \times (\xi_0 = -1)$  gets treated exactly like in 4) from Lemma 5, i.e. like in (21-C) and in Figure 7-bis. The point is that, when we add the  $b^3(\beta)$ 's to the Figures 9 which correspond to  $\Delta^2 \times (\xi_0 = -1)$  (and each of the two  $\beta$ 's in the Figure 19.1 corresponds to such a  $b^3(\beta)$ ), then this does not come (in the figures in question **with any 2<sup>d</sup> contribution from**  $\sum_1^M \mathcal{B}_i^2$  (see (79.1)). [Without the strategic decision from 4) Lemma 5, in a figure of type 24 for  $\Delta^2 \times (\xi_0 = -1)$  we could find transversal contacts like

$$B^2(\beta, z_+, y_+) \cap \{C(x_-, t_+) \subset \text{some curve } \Gamma_j\},$$

which would certainly contradict our 4). See here Figure 24, for a concrete illustration. BUT, with our strategic decision 4) in Lemma 5, illustrated in Figure 7-bis, at  $\xi_0 = -1$  the  $b^3(\beta)$  only comes with 1<sup>d</sup> attachements, no 2<sup>d</sup> ones, and the bad contact is avoided.]

With these things, clearly when  $\mathcal{B}_i^2 \subset \delta^2(b_i)$  is lifted off  $2X_0^2$ , then this happens far from  $\Delta^2 \times (\xi_0 = -1)$ . All the rest of  $\delta^2(b_i)$ , source of the map (80) is made out of spare parts  $d_k^2$ . When it comes to the construction of the  $\mathcal{B}_i^2 \subset \delta^2(b_i)$ , this can touch  $\Delta^2 \times (\xi_0 = -1)$  only via  $B^2(b_i)$ , along the edge  $e(b_i)$ . The  $B^2(b_i)$  is here like any other  $B_j^2$  in the inductive construction from (77.2) and (79-bis). We are here, with  $B^2(b_i)$  in the non-generic case. The edge  $e(b_i)$  comes with  $\Gamma(1) \times (\xi_0 = -1) \supset e(b_i) \subset c(b_i) \subset \partial B^2(b_i)$ . The  $\Delta^2 \times (\xi_0 = -1)$  with its adjacent  $B^2(b_i)$ 's, is presented in the Figure 19.1 which should help vizualise our discussion



**Figure 19.1.**

The interaction of  $\Delta^2 \times (\xi_0 = -1)$  with  $B^2(b_i) = [A, B, C, D, E, F, G, H]$ , with the interior of  $[E, F, G, H]$  being **deleted**.

**Additional explanations for Figure 19.1.** Without loss of generality, in the context of our figure, we have

$$JB^2(b_i) \cap (\Delta^2 \times (\xi_0 = -1)) = e(b_i).$$



At  $\xi_0 = 0$ , adjacent to our  $b_i$  lives the  $b^? = b_{j>i}$  and the question mark is there to suggest that (in the exceptional case), this  $b_j$  may be lacking. Anyway, the rectangle  $[ABCD]$  adjacent to  $b^?$  is like in Figure 7, unlike the lower  $[EFGH]$  which is like in Figure 7.bis. Every edge  $e \subset \Delta^2 \times (\xi_0 = -1)$  is like our present  $e(b_i) = [EH]$ , with a  $B^2(b_i)$  like in the Figure 19.1 adjacent to it. End of explanations.

**Important remark.** All the action in Lemma 10 is concentrated in

$$N^4(2X_0^2) \mid ((X_0^2 \times r) \cup (\Gamma(\infty) \times [r, b])). \quad \square$$

The next theorem and its proof will occupy the rest of this section IV.

**Theorem11. (Existence of external, embedded discs.)**

We can change the map (80) into a *DIFF* embedding, where not only  $J$  is changed but the  $\sum_1^M \partial \delta_i^2$  too

$$(82) \quad \left( \sum_{i=1}^M \delta_i^2, \sum_1^M \eta_i(\text{green}) = \partial \delta_i^2 \right) \xrightarrow{J = J \text{ (embedded)}} (\partial N^4(2X_0^2)^\wedge \times [0, 1] \subset N_1^4(2X_0^2)^\wedge, \partial N^4(2X_0^2)^\wedge \times \{0\}),$$

for which, eventually, we will have, in terms of the geometric intersection matrix

$$(82.1) \quad \eta_j(\text{green}) \cdot b_i^2 = \delta_{ji}, \text{ for } i, j \in \{1, \dots, M\}, \text{ AND } \eta_j(\text{green}) \cdot \left( B_1 - \sum_1^M b_i \right) = 0.$$

In other terms, we manage to create a system of external embedded discs (NO MORE ACCIDENTS), in exact cancelling position with  $\sum_1^M b_i$ .

[Here is what the “eventually” in the statement above means. In a first step, realized in this section IV already, we will realize a map

$$\left( \sum_1^M \delta_i^2, \sum_1^M \partial \delta_i^2 \equiv C_i(b)(\lambda) \right) \xrightarrow{J} (\partial N^4(2X_0^2)^\wedge \times [0, 1], \partial N^4(2X_0^2)^\wedge \times \{0\}),$$

which is completely ACCIDENT free (see (93.1) below), but without having yet the (82.1) satisfied. That will be eventually realized only by Theorem 13 in the next section V.]

**Remark.** As things stand, right now, there is no handlebody decomposition of  $N^4(\Delta^2)$  having the  $\sum_1^M b_i$  as its system of 1-handles. Here, notice that  $\Gamma(1) - \sum_1^M b_i$  is highly disconnected. This means that, Theorem 11, as it stands, cannot be plugged into Lemma 3, our (82.1) notwithstanding. Not to speak, also, of the fact that the  $\sum_1^M c(b_i)$  is not contained in  $\partial N^4(\Delta^2)$  (nor in  $N^4(\Delta^2)$ , for that matter).

**Proof.** Let's stay for the time being at level  $X^2(\text{old})$  and consider the purely spatial Figures 4. To them, we will add, in an appropriate way, the  $b^3(t_+)$  OR  $b^3(t_-)$  **never both simultaneously**. This leads to the Figures 9 where in addition to the already purely spatial smooth sheets, like  $(x_+, y_+, z_+)$  in Figure 4-B appear now new ones, like  $(x_+, y_+, t_\pm)$ , in the Figures 9-(B $_\pm$ ). Later  $\xi_0, \beta$  will occur too.

These figures are considered at the level  $X^2(\text{old})$ , for the time being.

The  $f_k d_k^2$  (76) consist of unions of local smooth sheets like the  $(x_{\pm}, y_{\pm}, z_{\pm})$ ,  $(x_{\pm}, y_{\pm}, t_{\pm})$  above, appearing with possible multiplicities, and not all of them may appear inside  $\bigcup_k f_k d_k^2$ . When we move to our enlarged figures from Figures 9, then the simple closed loops in those figures span little 2-cells contained in  $S^3(P) = \partial N^4(P)$ , like  $((x_{-}, t_{\pm}), (x_{+}))$  or  $((x_{-}, z_{+}), (x_{+}))$  in Figure 9-A or  $((z_{+}, t_{+}), (x_{-}))$  in Figure 9-B<sub>+</sub>. We will denote there 2-cells, generically by  $d^2(x_{-}, t_{\pm}, x_{+})$  (or, if a letter is without a surrounding circle, we put prentices around (like  $d^2(x_{-}, (z_{+}), x_{+})$ ). Since these  $d^2$ 's follow the rules of the game established by the paradigmatical Figure 8, we have the following

(83) At the level of the Figures 9, inside  $\partial N^4(P)$ , the various  $d^2$  (space-time, space-time, space-time) are 2-by-2 disjoint except for possible common edges or vertices.

Next we move from  $X^2(\text{old})$  to the realistic  $2X_0^2$ , where we will want to take  $J\delta^2$  off the  $\Delta^2 = \Delta^2 \times (\xi_0 = -1)$ . But before we can do that we will describe the changes to be operated at the level of the figures of type 9, when moving from  $X^2(\text{old})$  to  $2X_0^2$ .

(84.1) To our purely space-time figures 9, when we are at level  $X_0^2 \times r \subset 2X_0^2$ , we add more vertices, namely the following:

- ) A vertex  $(\beta)$  to **all** Figures 9 ( $\subset X_0^2 \times r$ ) corresponding to the axis  $\zeta$ , moving from  $r$  towards  $b$  (with, remember,  $r < \beta < b$  and  $|\beta - r| \ll |b - r|$ ). The  $[r, b]$  lives somewhere on the  $\zeta$ -axis.
- ) A vertex  $(+\xi_0)$  (respectively  $(-\xi_0)$ ) for every  $P \times (\xi_0 = -1)$ , when  $P \in \Gamma(1)$ , (respectively for every  $P \times (\xi_0 = 0)$ ,  $P \in \Gamma(1)$ ).

(84.2) This way, new arcs occuring in the REALISTIC FIGURES 9. Every Figure 9 acquires now a  $b^3(\beta) \subset B^3(+)$ . Any Figure 9 acquires a  $b^3(-\xi_0) \subset B^3(-)$ , if it lives at  $\xi_0 = 0$ , respectively a  $b^3(+\xi_0) \subset B^3(-)$ , if it lives at  $\xi_0 = -1$ . Here are the **golden rules** for drawing our realistic Figures 9.

•••) For  $P \times (\xi_0 = -1) \subset \Delta^2 = \Delta^2 \times (\xi_0 = -1)$ , never contradict what is already done in the Figures 9. The “Figures 9” are all compatible with the paradigmatical Figure 8, so the present rule makes (22.A) automatically satisfied, provided we proceed like in  $X^2(\text{old})$ , along the edges of  $\Delta^2 \times (\xi_0 = -1)$ , in particular when the framing for attaching 2-handles are concerned.

••••) In terms of the basic splitting from (27.6), (28), the Figures 9 live essentially in  $B^3(-)$  (to be more precise in  $B^3(-) \cup B^3(\text{in})$ ), while we have  $b^3(\beta) \subset B^3(+)(\cup B^3(\text{out}))$ . Figure 9-bis gives the accurate picture.

•••••) For our  $P \times (\xi_0 = 0)$  or  $P \times (\xi_0 = -1)$ , the  $b^3(-\xi_0)$ , respectively  $b^3(+\xi_0)$ , live inside  $B^3(-)$ , like the rest of Figures 9 and, if we think that  $t_{+}$  points towards the observer and  $t_{-}$  away from the observer, then  $b^3(+\xi_0)$  lives higher than  $b^3(t_{+})$  and  $b^3(-\xi_0)$  lower than  $b^3(t_{-})$ . The  $b^3(t_{+})/b^3(t_{-})$  are mirror images of each other, a condition stemming from Figures 8 and 8-(B).

Now the  $\beta$  and the  $\pm \xi_0$  have nothing to do with  $R^4$  (which imposes Figure 8) and no metric restrictions concern them for our 4<sup>d</sup> reconstruction formulae. But then, for reasons of simplicity, we will impose on  $b^3(-\xi_0)/b^3(+\xi_0)$  the same kind of mirror symmetry as for  $b^3(t_{-})/b^3(t_{+})$ . Other better reasons for this might become clear later on.

[The mirroring is like in the Figure 8-(B) BUT, careful, do not mix up the splittings  $B^3(\text{out}) \cup \underbrace{B^3(\text{in})}_{S^2(P)}$  and  $B^3(+)\underbrace{\cup}_{S_{\infty}^2} B^3(-)$ . The  $S^2(P)$  is defined by Figure 8-(C) and the relation  $S^2(P)/S_{\infty}^2$  is shown in Figure 9.bis. Approximatively, one may think in terms of  $B^3(\text{in}) \subset B^3(-)$ .]

•••••) Any crossing of lines involving  $(\beta)$ ,  $(\pm\xi_0)$  or space-time lines **not** at  $\Delta^2 \times (\xi_0 = -1)$  are **acceptable**, as long as (22-B) is also satisfied. We will call this the **crossing freedom**. This comes with **local changes in topology**. But, once  $C \cdot h = \text{id} + \text{nil}$  stays with us, the global topology stays unchanged.

•••••) We do insist that (83) should be satisfied everywhere. [As a consequence of b), e) in the definition (5), in any figure of type 24 there is **at most one** crossing ( $\times$ ) of space-time lines. Any change of **this** one by crossing UP  $\Leftrightarrow$  DOWN of the two lines, cannot be about any violation of (83). And then, our (83) does not concern neither  $\beta$  nor  $\pm\xi_0$ .] End of (84.2)

(85) When it comes to the realistic  $f_k d_k$  (76) it is only the  $P \in X_0^2 \times r$  which are involved, and in the corresponding realistic Figures 9,  $(\beta)$ 's are mute (for  $d_k$  but **not** for  $\delta_k^2$ ).

**Lemma 11.1.** 1) We can lift the map  $\sum_1^N d_k^2 \xrightarrow{\Sigma f_k} X_0^2 \times r \subset 2X_0^2$  to an immersion winding tightly around  $X_0^2 \subset N_1^4(2X_0^2)^\wedge$

$$(86) \quad \sum_1^N d_k^2 \xrightarrow{J} N_1^4(2X_0^2)^\wedge,$$

with ACCIDENTS like in 3) Lemma 10. This is part of (80).

2) We can get our LIFT (86) by the following two steps process. One starts by lifting  $\sum_1^N d_k^2$  to a **not everywhere well-defined** map

$$(87) \quad \sum_1^N d_k^2 \xrightarrow{F} \partial N^4(2X_0^2),$$

and for describing it one needs a partition, to be made explicit below:  $d_k^2 = \text{body } d_k^2 + (d_k^2 - \text{body } d_k^2)$ . Here  $F \mid (d_k^2 - \text{body } d_k^2)$  is completely well-defined everywhere and WITHOUT ACCIDENTS. It runs parallel and very close to the  $D^2(\Gamma_j \text{ or } C_i) \subset X_0^2 \times r$ . More precisely, for each of the  $D^2(\Gamma_j \text{ or } C_i)$  which is involved, the corresponding piece of  $F(d_k^2 - \text{body } d_k^2)$  lives on the lateral surface of the corresponding  $4^d$  handle of index  $\lambda = 2$ , call it  $S^1 \times D^2$ , taking the form  $(*) \times D^2$ .

So, the disc  $d_k^2$  is broken into a  $\{\text{disc with holes, called body } d_k^2\}$  of boundary  $\{\gamma_k^0 + (\text{internal circles})\}$ , while  $d_k^2 - \text{body } d_k^2$  is a disjointed collection of discs  $D^2(C), D^2(\Gamma)$ , filling the internal circles in question.

3) It is the map

$$(88) \quad \sum_1^N \text{body } d_k^2 \xrightarrow{F} \partial N^4(\Gamma_1(\infty))$$

which is not everywhere well-defined, since it comes with **punctures**, i.e. transversal contacts

$$(88.1) \quad F(\text{body } d_k) \cap \{\text{attaching zones of the 2-handles, i.e. the} \\ \{\text{curves } \Gamma_j \text{ or } C_i\} \times b^2(\text{framing}) \subset \partial N^4(2\Gamma(\infty))\}.$$

These punctures as well as the double points  $M^2(F)$  create the ACCIDENTS of the lifted map (86). Before going on with the (86) we will open a

**Prentice.** We have a decomposition

$$(88.2) \quad \sum_1^N d_k^2 = \left\{ \sum_1^N d_k^2 \mid X^2(\text{old}) \text{ from which we exclude the interiors of the 2-cells in } d_k^2 \cap (\Delta^2 \times (\xi_0 = 0)) \right\} \\ + \{ \text{a contribution from } (\Gamma(1) \times [0 \geq \xi_0 \geq -1]) \cup (\Delta^2 \times (\xi_0 = -1)) \text{ which requires the deletion just done} \}.$$

More explicitly, we change  $d_k^2 \mid X^2(\text{old})$  by replacing every  $\sigma^2 \subset (d_k^2 \mid X^2(\text{old})) \cap (\Delta^2 \times (\xi_0 = 0))$  by  $(\partial\sigma^2 \times [0 \geq \xi_0 \geq -1]) \cup (\sigma^2 \times (\xi_0 = -1))$ .

Once the contribution of  $\Delta^2 \times (\xi_0 = 0)$  has been removed, the rest of body  $\left( \sum_1^N d_k^2 \mid X^2(\text{old}) \right)$  stays intact, inside  $\partial N^4(\Gamma_1(\infty))$  (see here  $\bullet\bullet\bullet\bullet$  in (84.2)).

Because of  $\bullet$  in (84.2) and because we never mock around with the Figures 9 at  $(\xi_0 = -1)$ , the body  $\left( \sum_1^N d_k^2 \mid (\Delta^2 \times (\xi_0 = -1)) \right)$  comes without any special problems. The body  $\left( \sum_1^N d_k^2 \mid (\Gamma(1) \times [0 \geq \xi_0 \geq -1]) \right)$  is a topic to be discussed more in detail, below. End of prentice.

*All this ends a first part of our road to (86) and the second part will consist in the {lifting of (88) off from  $\partial N^4(\Gamma_1(\infty))$ } + {the much more trivial step of lifting the  $F \mid (d_k^2 - \text{body } d_k^2)$ }. This last piece consists just of parallel copies of the various  $D^2(C_i)$  or  $D^2(\Gamma_j)$ . As already said, it is the lift of (88) which creates the ACCIDENTS of (86).*

*A priori, the punctures in (88.1) are accompanied by CLASPS and RIBBONS. Now, at the level of the embellished Figures 9, if the  $b^3(\pm\xi_0)$  are lacking (and see here Figure 24 for illustration, with all the red part deleted) and if we, moreover, restrict ourselves to the  $\sum_1^N d_k^2$  part of  $\sum_1^M \delta_i^2$ , and hence forget the  $b^3(\beta)$  (and hence the green part of Figure 24 too), then it follows from (83) that there are **no transversal contacts***

$$d^2(\text{space-time, space-time, space-time}) \cap C(\text{space-time}).$$

*But then, we will have **transversal contacts***

$$d^2(\pm\xi_0, \text{space-time, space-time}) \cap C(\text{space-time}),$$

*when “C” may mean actual  $C_i$  or  $\Gamma_j$ . There will be made explicit, completely, and they produce both the (88.1) and the*

$$M^2 \left( F \mid \sum_{k=1}^N \text{body } d_k^2 \right).$$

*We will also have*

$$M^3 \left( F \mid \sum_{k=1}^N \text{body } d_k^2 \right) = \emptyset = M^3 \left( J \mid \sum_1^N d_k^2 \right).$$

*This absence of triple points follows because, at the level of the figures of type 24, the only conceivable way to produce some  $M^3$  would be*

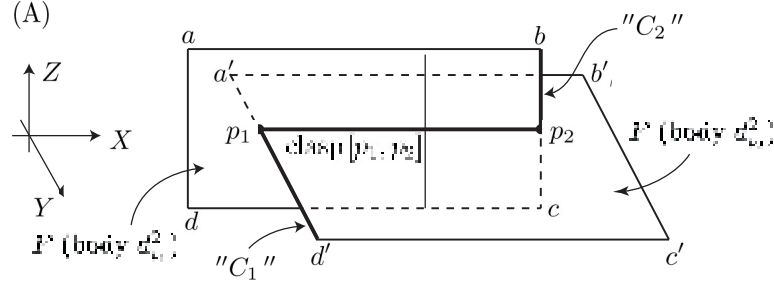
$$(*) \quad d^2(\pm\xi_0, \text{space-time, space-time}) \cap d^2(\text{pure space-time}) \cap d^2(\text{pure space-time})$$

*and, once  $d^2(\text{pure space-time}) \cap d^2(\text{space-time}) = \emptyset$ , the  $(*)$  is also absent.*

*When we are **far from**  $\Delta^2 \times [0 \geq \xi_0 \geq -1]$ , there are neither CLASPS nor RIBBONS for  $\sum_1^N d_k^2$ . Any CLASP has to involve  $d^2(\pm\xi_0, \text{space-time})$ . Moreover, a direct analysis will show that the CLASPS, occurring all of them in the region  $\Delta^2 \times [0 \geq \xi_0 \geq -1]$ , are 2-by-2 disjointed.*

*Figure 20 gives a general illustration of the CLASPS of  $\sum_k d_k^2$  and Figure 21 of the RIBBONS.*

So, our Figure 20 which lives inside  $\partial N^4(\Gamma_1(\infty))$  illustrates a clasp of (88) and, a more detailed view is provided by the Figure 22.



This figure is in  $\partial N^4(\Gamma_1(\infty))$ . The  $C_1 = [a', p_1, b']$  and  $C_2 = [c, p_2, b]$  are pieces of two curves  $\Gamma$  or  $C$ , attaching zones of 2-handles. The  $p_1, p_2$  are punctures, like in (88.1). Inside the  $\partial N^4(2\Gamma_1(\infty))$ , the pieces of  $F(\text{body } d_k^2)$  seable in this figure, continue beyond the thin contours  $[ba] \cup [ad] \cup [dc]$  and  $[a'b'] \cup [b'c'] \cup [c'd']$ .

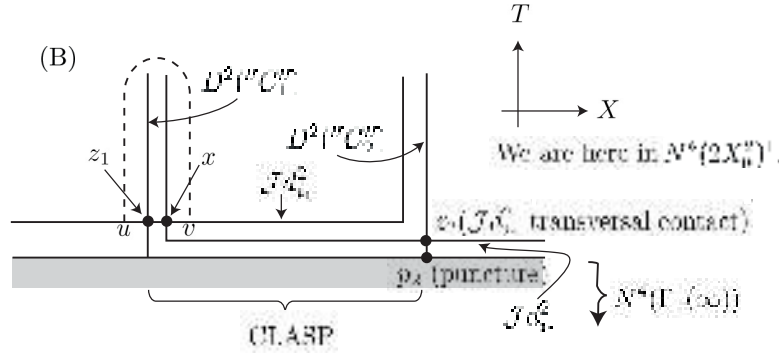


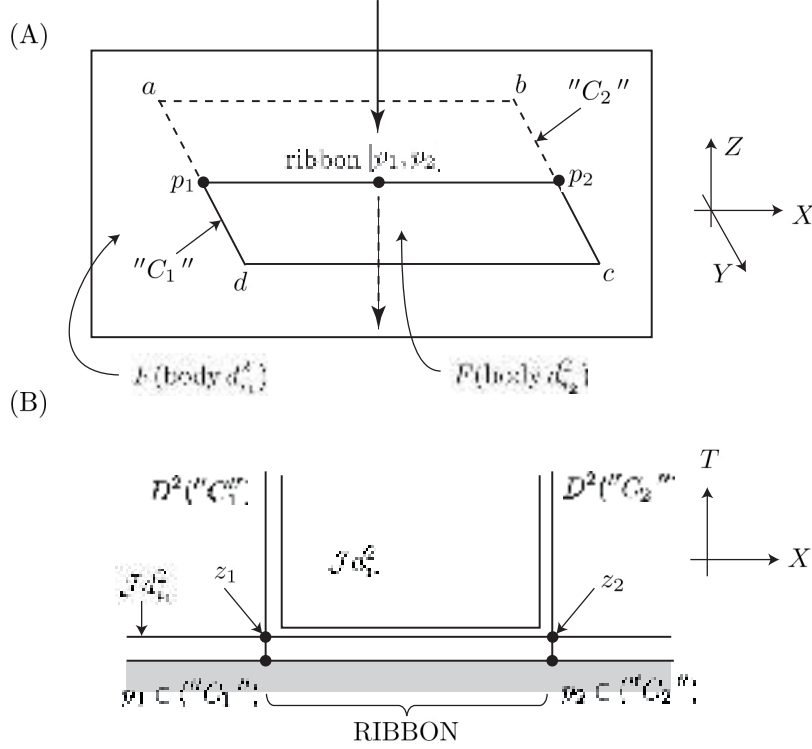
Figure 20.

(A) is generic figure for a CLASP of (88), creating in  $4^d$  (in (B)) the ACCIDENTS  $z_1, z_2$  and  $x$ . Figure 23 is a more accurate and detailed version of (B). The process of pushing over the  $(\text{extended cocore})^\wedge$  of  $z_1$ , is suggested in dotted lines.

Here are **some explanations concerning Figure 20**. Beyond the contours  $[b, a] \cup [a, d] \cup [d, c]$  and  $[a', b'] \cup [b', c'] \cup [c', d']$ , body  $d_k^2$ 's continues, while at the " $C_1$ " and " $C_2$ ", where the  $d_k^2$ 's climb on the corresponding  $D^2$ ("C")'s, the bodies stop. The (B) shows a lift of the  $d_{i1}^2, d_{i2}^2$  from (A) into  $N_1^4(2X_0^2)^\wedge$ , cut by the plane  $(X, T)$ ; this is also transversal to  $\partial N^4(\Gamma_1(\infty))$ , the space of Figure (A). The  $[u, v]$  in (B) is the trace of a small disc  $D^2 \subset Jd_{i1}^2$ , centered at the transversal contact  $z_1$  and cutting transversally, both through  $D^2$ ("C1") and to  $Jd_{i2}^2$ . In the more specific Figure 23, which will supersede the present (B), this disc  $D^2$  is put to work. Importantly, the double point  $x$  is contained in  $D^2$  too. Here  $d_{i1}^2 \subset \{\text{some } \delta_i^2\}$  and, the  $J \sum_1^M \delta_i^2$  runs close, at some distance  $\varepsilon > 0$  from  $\sum_1^M g_i \delta_i^2 \subset 2X_0^2$  (see (79)).

We have here  $\text{diam } D^2 \gg \varepsilon$ , which will make that there is no obstruction for the operation of pushing  $Jd_{i1}^2$  OVER the  $\{\text{extended cocore } z_1\}^\wedge$  (when it exists), like in the Figure 23.

We continue with



**Figure 21.**

A RIBBON of the map (88). The vertical arrow, crossing the RIBBON is part of a green arc.

**More explanations for the Figures 20 and 21.** One goes from (88) to the corresponding part of (86), by replacing each  $F(\text{body } d_k^2)$  by  $F(\text{body } d_k^2) \times [0, \varepsilon_k]$  where  $F(\text{body } d_k^2) \times \{0\} = F(\text{body } d_k^2)$  and the  $[0, \varepsilon_k]$  is outgoing from  $\partial N^4(2X_0^2)$  (direction  $[0, 1] \subset \text{coordinate } T$ ). The  $\varepsilon_k$ , depending on  $k$  is a **free choice** which we can make, normally at least, as we please, according to our convenience. Whatever this choice is, at the level of Figure 20-(B) (CLASP), there **has** to be a double point  $x \in JM^2(J)$ . But then, in Figure 20-(B) we did make a **choice**  $\varepsilon_{i_2} > \varepsilon_{i_1}$  ( $> 0$ ) and **if**  $z_1$  **possesses an extended cocore** ( $z_1$ ) (which is exactly the case when our “ $C_1$ ” is a  $C_i$  and **not** a  $\Gamma_j$ ), then as we have suggested in Figure 23 in dotted lines, there is now a procedure for **killing both**  $z_1$  **and**  $x$ , by pushing  $Jd_{i_1}^2$  over the  $\{\text{extended cocore } (z_1)\}^\wedge$ . BUT, even if  $z_2$  also possesses an extended cocore ( $z_2$ ), we still cannot apply this procedure both for  $z_1$  and  $z_2$ . The so-called **“green arc process”** will be necessary then.

Figure 21-(B) plays the same role with respect to 21-(A) as the 20-(B) with respect to 20-(A). But now, with the choice  $\varepsilon_{i_2} > \varepsilon_{i_1}$  which be made (in Figure 21-(B)), there are no double points  $x \in JM^2(J)$  attached to the ribbon. With the choice  $\varepsilon_{i_2} < \varepsilon_{i_1}$  there are two such. A priori, both choices are locally possible and it is the global policy of getting rid of accidents, AND of CONFINING the green arcs inside  $\partial N_+^4(\Gamma_1(\infty))$ , which will determine which choices are appropriate and which are not. Notice, also, that while in Figure 20-(B) we have  $D^2(\text{“}C_2\text{”}) \subset (Y, T)$ ,  $D^2(\text{“}C_1\text{”}) \subset (Z, T)$ , in 21-(B) we find

$$D^2(\text{“}C_1\text{”}) \subset (Y, T) \supset D^2(\text{“}C_2\text{”}).$$

End of explanations.  $\square$

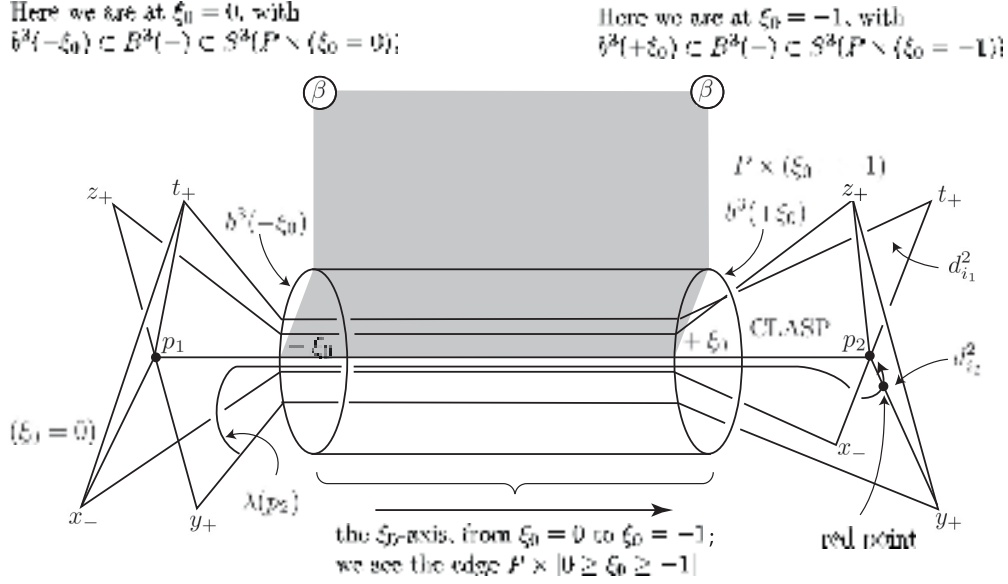


Figure 22.

We see here in the NORMAL situation, the only clasp which (88) may have, occurring between the pair of adjacent vertices  $P \times (\xi_0 = 0)$  and  $P \times (\xi_0 = -1)$ , with the same  $P \in \Gamma(1) \subset \Delta^2$ . Here to begin with, at  $\xi_0 = -1$  for illustration we have Figure 9-(B<sub>+</sub>) for  $P \times (\xi_0 = -1)$ , with a  $b^3(+\xi_0)$  (which gets added when we go to  $X_0^2[\text{new}] \subset 2X_0^2$ ), living closer, visually, to the observer, i.e. higher, than everything else, seable in the Figures 9. Also, with complete details, the fully embellished Figure 9-(B<sub>+</sub>) occurs as in Figure 24-(A). This accounts for the situation at  $p_2 \in \Delta^2 \times (\xi_0 = -1)$ . The situation at  $\xi_0 = 0$  is to be discussed in the main text. A priori, making use of our free choices, it would be possible to change this CLASP into a RIBBON. But we could not live with both CLASPS and RIBBONS along the various  $P \times [0 \geq \xi_0 \geq -1]$ . So, our free choices at  $\xi_0 = 0$ , will be used to have consistently only CLASPS.

LEGEND: — =  $d^2(\text{space-time, space-time, space-time})$ ;

→ = green arc  $\lambda(p_2)$ . Here  $\lambda(p_2) \subset \{\text{the } [y_+, z_+] \text{ branch which cuts transversally the } \Gamma_j \text{ at } p_2\}$  (if we think of it inside body  $d^2$ ). The red intersection point

$$\lambda(p_2) \cap \{\text{RIBBON starting at } p_2\}$$

is harmless, since the ribbon in question will receive the treatment of Figure 28-bis. In particular, at the RHS of our figure, we have  $\lambda(p_2) \subset d_{i_2}^2$ , coming with  $\varepsilon(d_{i_2}^2) < \varepsilon(d_{i_3}^2)$  ( $d_{i_3}^2 \equiv$  the branch of the RIBBON  $p_2 \rightarrow y_+$ ). So, the lifted  $\lambda(p_2)$  flows under the lifted  $d_{i_3}^2$ .

4) We will make use of the following **crossing-freedom** when we will put up the completely embellished Figures 9. To begin with, edges starting at  $b^3(\pm \xi_0)$  should **never** be cut by triangles  $d^2(\text{space-time, space-time, space-time})$ . Moreover, edges starting at  $b^3(\beta)$  should not be cut by anything at all. Figure 24-(A) should illustrate these things.

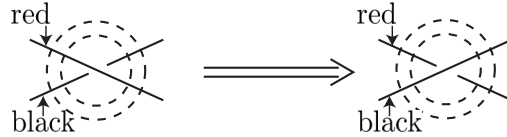
With this, all the CLASPS and RIBBONS of (88), source of the accidents (86), are of the following types.

[And retain also that, since the  $b^3(\beta)$  is present only for the part  $B_i^2 \subset \delta_i^2$ , the (87), (88), which only concern the  $d_k^2$ -parts, do not contain any  $\beta$ -contribution.]

4.1) CLASPS internal to  $\Delta^2 \times [0 \geq \xi_0 \geq -1]$ , occurring exactly along SOME  $P \times [0 \geq \xi_0 \geq -1]$ 's. Here is the mechanism which creates these CLASPS. To begin with, as a consequence of our constant use of the GPS structures, in any given Figure 9 **there is at most one crossing of space-time lines**. With this, consider for illustration the crossing  $[y_+, z_+](\text{DOWN})/[x_-, t_+](\text{UP})$  in Figure 9-(B<sub>+</sub>) and occurring also in the RHS of Figure 22. At this point, remember that at  $\xi_0 = -1$  we have to add an axis  $+\xi_0$  in addition to space-time and at  $\xi_0 = 0$  an axis  $-\xi_0$ .

For  $+\xi_0$  we have Figure 24-(A) as it stands. When it comes to the axis  $-\xi_0$  at  $\xi_0 = 0$ , here is what goes on. We certainly have our crossing  $[y_+, z_+](\text{DOWN})/[x_-, t_+](\text{UP})$ , which **stays put**, and for  $b^3(-\xi_0)$  we chose a position which is the **MIRROR IMAGE** with respect to  $S^2(P)$  (see Figure 8-(C)), of the position of  $b^3(+\xi_0)$  in Figure 24-(A), like for  $t_{\pm}$  in Figure 8-(A).

So  $b^3(-\xi_0)$  (contrary to  $b^3(+\xi_0)$  in Figure 24-(A)) is now lower with respect with anything in space-time. In terms of Figure 24-(A) this amounts to the following changes: i) a notational change  $+\xi_0 \Rightarrow -\xi_0$ , ii) an allowed crossing, performed at each site with a circle drawn as a broken dark line, performed at the level of Figure 24,



with RED =  $b^3(-\xi_0)$ -line, BLACK = space-time line.

It can be checked that this produces the CLASP from Figure 22.

4.2) All the RIBBONS of (88) occur at  $\xi_0 = -1$  AND at  $\xi_0 = 0$ . In terms of Figure 22, RIBBONS at  $\xi_0 = -1$  go along two neighbouring  $p_2$ 's. They corresponds to things like  $[p_2, z_+]$ ,  $[p_2, y_+]$ . Dually, RIBBONS at  $\xi_0 = 0$  go along two neighbouring  $p_1$ 's and, again, they correspond to things like  $[p_1, t_+]$ ,  $[p_1, x_-]$ . So, all the accidents of (88) are concentrated at  $\Delta^2 \times [0 \geq \xi_0 \geq -1]$ . End of Lemma 11.1.

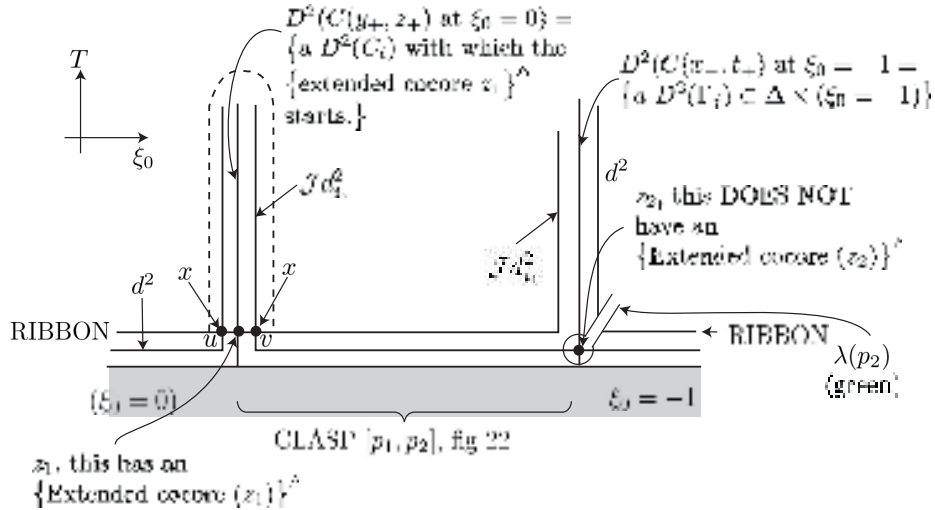


Figure 23.

The real-life effect of Figure 22. All the ACCIDENTS of  $Jd^2$  (86) occur like in this figure, for the various clasps along  $[0 \geq \xi_0 \geq -1]$ , which are 2-by-2 disjointed, and see here 4.1) in Lemma 11.1, and also 4.2) from the same lemma. The RIBBONS at  $\xi_0 = -1$ , respectively at  $\xi_0 = 0$ , form each of them, a system which, IF each  $P \times [0 \geq \xi_0 \geq -1]$  would carry a clasp like in Figure 22 would be connected. But, in real life, since there are not necessarily CLASPS along each



$P \times [0 \geq \xi_0 \geq -1]$ , this breaks into several connected systems, each of them possibly highly non-simply connected. The clasps along  $[p_1, p_2]$  connect the two multi-systems.

In our present figure, at  $p_1$  both  $z_1$  and the  $x$ 's can be demolished by pushing  $Jd_{i_1}^2$  over the  $\{\text{extended cocore } (z_1)\}^\wedge$ . Of course, at  $p_2$ , such a cocore is not available. But notice that, even if it would be, something else would still be needed. One cannot use a push over the  $\{\text{extended cocores}\}^\wedge$ , at **both** ends of a CLASP, even when such extended cocores do exist.

**Remarks.** A) In order to define our  $\sum_1^M \delta_i^2$ , we need the BLUE order of  $X^2[\text{new}]$ , and not the BLUE order of  $2X^2$ . In the context of  $2X^2$ , on the side  $X_0^2 \times r = X_0^2[\text{new}]$ , all the  $2^d$  cells are  $D^2(\gamma^1)$ 's BLUE-wise, with the real BLUE order transferred on the  $b$ -side. To define the  $\delta_i^2$ 's, we need the DOUBLE  $2X_0^2$ , in particular the (77.2)'s. Also, in this same context, all the  $B_i^2$ 's occurring inside  $\sum_1^M \delta_i^2$  are exactly the  $B_j^2$ 's produced by the BLUE  $X^2[\text{new}]$ -trajectories of the  $\sum_1^M b_i$ ; see (77.1).

B) So far, we have only looked at the RIBBONS body  $d^2 \pitchfork \text{body } d^2$ , but there will also be RIBBONS  $\{\text{body } \mathcal{B}^2 \text{ (long branch)} \pitchfork \text{body } d^2 \text{ (short branch)}\}$ .

C) When it comes to destroying the ACCIDENTS, it will be convenient to have only CLASPS along the  $P \times [0 \geq \xi_0 \geq -1]$  and NO RIBBONS going from  $P \times (\xi_0 = 0)$  to  $P \times (\xi_0 = -1)$ . End of Remarks.

Each vertex  $P \in 2X_0^2$  comes with an “embellished Figure 9”, actually Figure 24 and the purely space-time part of these figures (that is what one actually sees in Figures 9), lives inside  $B^3(-)(P)$ . But, of course, the complete, real-life, embellished Figures 9 contain the additional items too, namely the following.

- A  $b^3(\beta) \subset B^3(+)$ , present for all  $P$ 's.
- A  $b^3(+\xi_0 \text{ or } -\xi_0) \subset B^3(-)$  for  $P \times (\xi_0 = -1 \text{ or } \xi_0 = 0)$ . At  $\xi_0 = -1$  this is placed higher than the space-time items, and at  $\xi_0 = 0$  lower.
- In any of the embellished Figures 9 one has to add edges  $b^3(\beta) \text{ — } b^3(\text{space-time})$  and at  $\{\xi_0 = -1, \xi_0 = 0\}$  one also adds edges  $b^3(\pm \xi_0) \text{ — } b^3(\beta \text{ or space-time})$ .
- We have triangles  $d^2(\pm \xi_0, \text{space-time}, \text{space-time})$ ,  $B^2(\beta, \pm \xi_0, \text{space-time})$ ,  $B^2(\beta, \text{space-time}, \text{space-time})$  in addition to any purely space-time triangle in Figures 9. At  $\xi_0 = -1$ , the interiors of the triangles  $B^2(\beta, \dots, \dots)$  are **void**, like in the Figures 7-bis and 19.1.

Using the **crossing-freedom**, we may impose the following

(90) No triangle  $d^2(\dots, \dots, \dots)$  at all cuts through an edge starting at  $b^3(\beta)$ . Here  $\dots$  stands for  $\{\pm \xi_0 \text{ or space-time}\}$ . So, there are no CLASPS containing  $b^3(\beta)$  and, when Figure 21 is not connected with  $\Delta^2 \times [0 \geq \xi_0 \geq -1]$ , then we have body “ $d_{i_1}^2$ ”  $\subset \mathcal{B}^2$ , and the curve it goes through transversally is  $C$  (space-time, space-time) OR  $C(\pm \xi_0, \text{space-time})$  (see, for this last item the doubly circled crossings in Figure 24).

(91) There are NO TRIPLE POINTS FOR the map  $J$  in (80), i.e.

$$M^3 \left( J \mid \sum_1^M \delta_i^2 \right) = \emptyset.$$

[Proof of (91). We know this already for the restriction  $J \mid \sum_k d_k^2$ . With everything we have already said, the triple points could only be produced by something of the following type, with possibly the  $d^2(\text{space-time})$  replaced by another  $B^2(\beta, \pm \xi_0$  and/or space-time)

$$d^2(\pm \xi_0, \text{space-time}) \cap d^2(\text{space-time}) \cap B^2(\beta, \pm \xi_0 \text{ and/or space-time}).$$

In order to conjure these out of existence we will make use, according to the case, either of the STRATEGIC DECISION 4) in the Lemma 5, which comes with the Figures 7-bis and 19.1, OR of the crossing freedom when outside of  $\Delta^2 = \Delta^2 \times (\xi_0 = -1)$ .

With  $b^3(+\xi_0)$ , at  $P \times (\xi_0 = -1)$ , such triple points do not exist, because the  $B^2$ -triangles have now no interior; see our Figures 7-bis and 19.1. So we look at  $P \times (\xi_0 = 0)$ , and now  $b^3(-\xi_0)$  is the MIRROR-IMAGE of the  $b^3(+\xi_0)$  from Figure 24-(A). The  $b^3(\beta)$ -lines continue to be “over the table”, just like in Figure 24, while now the  $b^3(-\xi_0)$ -lines are “under the table”.

With this, the double-line  $d^2(-\xi_0, x_-, t_+) \cap d^2(-\xi_0, z_+, y_+)$  (and there are no other lines  $d(-\xi_0, \text{space-time}, \text{space-time}) \cap d^2(-\xi_0, \text{space-time}, \text{space-time})$ ) cannot be cut by any  $B^2(\beta, \text{space-time}, \text{space-time})$ . And it can be seen, directly, that it is not cut by  $B^2(\beta, -\xi_0, \text{space-time})$  either.

This proves our (91).  $\square$

**Complement to Lemma 11.1.** *We can extend Lemma 11.1 from  $\sum d_k^2$  to the whole of  $\sum \delta_i^2$ , where*

$$\delta_i^2 = \mathcal{B}_i^2 \cup \sum d_k^2 \text{'s, see (79.1).}$$

*To begin with, there is a decomposition*

$$\mathcal{B}_i^2 = \text{body } \mathcal{B}_i^2 + (\mathcal{B}_i^2 - \text{body } \mathcal{B}_i^2),$$

*with a not everywhere well-defined map like in (87)*

$$(91) \quad \sum_1^n \mathcal{B}_i^2 \xrightarrow{F} \partial N^4(2X_0^2),$$

*coming with the analogue of (88), namely with*

$$(92) \quad \sum_1^n \text{body } \mathcal{B}_i^2 \xrightarrow{F} \partial N^4(\Gamma_1(\infty)),$$

*with  $F \mid (\mathcal{B}_i^2 - \text{body } \mathcal{B}_i^2)$  very much like the  $F \mid (d_k^2 - \text{body } d_k^2)$  at 2) in Lemma 11.1.*

1) *Once nothing cuts through edges of  $\partial N^4(P)$  having  $b^3(\beta)$  at one end (see Figure 24), all the punctures of (92) concern curves  $C(\pm \xi_0, \text{space-time})$  or  $C(\text{space-time}, \text{space-time})$ . These come with RIBBONS only, more precisely*

$$\text{RIBBONS}\{\text{body } \mathcal{B}^2\}(\text{long branch}) \cap \{\text{body } d^2\}(\text{short branch}) \text{ OR RIBBONS}\{\text{body } \mathcal{B}^2\} \cap \{\text{body } \mathcal{B}^2\}.$$

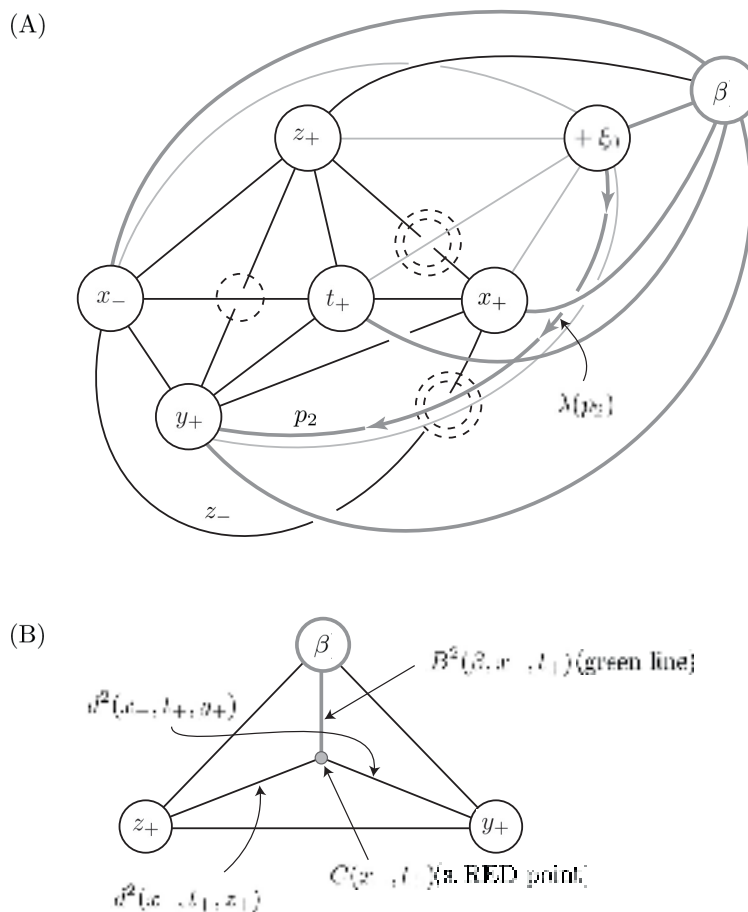
*All the double points and the transversal contacts when (91) is involved, come from the RIBBONS of (92) (which has no other double points to which it participates).*

2) *When we go to  $(J(80)) \mid \sum_1^M \mathcal{B}_i^2$ , this comes with the following ACCIDENTS (part of 3) Lemma 10): double points  $x \in J\mathcal{B}^2 \cap Jd^2$  and transversal contacts  $z \in J\mathcal{B}^2 \cap (2X_0^2 - \Delta^2 \times (\xi_0 = -1))$ .*

*To these, a priori, one might have to add double point  $x \in J\mathcal{B}^2 \cap J\mathcal{B}^2$  produced by the internal RIBBONS of  $\{\text{body } \mathcal{B}^2\}$ . When the detailed treatment of all the RIBBONS will have been reviewed we will see what happens with these potential double points. End of the complement to Lemma 11.1.*

**Remark.** There is a good a priori reason why  $M^2\left(J \mid \sum_1^M \mathcal{B}_i^2\right) = \emptyset$ . Inside each individual Figure 24 (i.e. for each individual  $\partial N^4(P)$ ), there is a unique  $b^3(\beta)$  spanning a number of distinct  $B^2(\beta, \dots)$ -triangles which, except for common edges (and the common vertex at  $b^3(\beta)$ ) are disjoint. So, the only potential source for  $M_2\left(J \mid \sum_1^M \mathcal{B}_i^2\right)$  is dry. End of Remark.

What we see now at the level of the Figure 24 below and its likes, happens at the level of (88) and (92). This figure is actually a fully embellished version of Figure 9- $(B_+)$ . It exhibits triangles of types  $d^2$  and  $\mathcal{B}^2$ , all being part of the corresponding body (...)’s.



LEGEND  $\rightarrow$  — = green arc  $\lambda(p_2)$ . This is NOT an edge, part of  $C_1, \Gamma_j$ .  
 — = edge  $[\epsilon_0, \text{space-time}]$ . This is red.  
 — = edge  $[\beta, \text{space-time OR } + \epsilon_0]$ . This is green.

Figure 24.

In (A) we present the completely embellished Figure 9- $(B_+)$ , in the context of  $P \times (\xi_0 = -1) \in \Delta^2$ . Then the  $B^2$ -triangles have a void interior (see here the Figure 22).

The Figure (B) concerns the context of  $P \times (\xi_0 = 0)$  and it presents the  $B^2$ -triangle  $B^2(\beta, y_+, z_+)$ . If this  $B^2$  would be with its interior alive, present at  $P \times (\xi_0 = -1)$ , we would see exactly the

same picture; when we go from  $\xi_0 = -1$  to  $\xi_0 = 0$ , it is only  $+\xi_0$  which flips to the mirror-position  $-\xi_0$ .]

Here are **some explanations concerning Figure 24**. The **edges** are occurring, according to the case in black (= pure space-time), red (= one vertex  $\pm \xi_0$  but no  $\beta$ ), green (= one vertex  $\beta$ ), BUT the green arc marked  $\lambda(p_2)$  is **not** such an edge. The green edges do not cut transversally (like  $R \cap R^2 \subset R^3$ ) through the space-time triangles  $d^2(\text{space-time, space-time, space-time})$ . The (A) lives in  $S^3(p) = B^3(-) \underset{S_\infty^2}{\cup} B^3(+)$ .

It completes Figure 9- $(B_+)$ , and here is how to read it, starting from the Figure 9- $(B_+)$ , in the situation  $P \times (\xi_0 = -1)$ , or  $P \times (\xi_0 = 0)$ . Add, inside this same  $B^3(-)$ , the  $b^3(+\xi_0)$  higher than anything else (i.e. closer to the observer). This gives an embellished Figure 9- $(B_+)$  which we will flatten (i.e. the space-time,  $+\xi_0$  lines) like in a link diagram, with crossings UP/DOWN. Next one adds the  $b^3(\beta) \subset B^3(+)$  and suspend from it, with green lines, the now conveniently flattened {embellished Figure 9- $(B_+)$ }.

This makes sure that nothing can cut transversally through the  $\beta$ -lines.

All this should help vizualizing things. Incidentally, I am using here the word “suspending” in a rather cavalier manner; I should have said “coning”, instead. At  $P \times (\xi_0 = -1)$  the interiors of the  $B^2$ -triangles are removed, and we turn to  $P \times (\xi_0 = 0)$ . Then our  $+\xi_0$  becomes  $-\xi_0$ , in a MIRROR-IMAGE POSITION, with respect to what we see in Figure 24-(A).

One may notice that the  $B^2$ -triangles never cut the lines  $C(\pm \xi_0, \text{space-time})$ . [We only have to consider  $P \times (\xi_0 = 0)$  with  $b^3(-\xi_0)$ . With  $b^3(-\xi_0)$  in its MIRROR IMAGE location, “under the table”, we clearly do not have

$$B^2(\beta, \dots) \cap C(-\xi_0, \text{space-time}).]$$

Figure (B), with a red transversal contact  $B^2(\beta, \dots) \cap C(\text{space-time, space-time})$ , corresponds to the unique crossing of black lines, surrounded by a dotted loop, in (A). And here again, is the move (A)  $\Rightarrow$  (B). Imagine the BLACK and RED lines in (A) displayed “on the table”, like in a link diagram. The green  $b^3(\beta)$  lives at infinity and the  $B^2(\beta, \dots)$  triangles, like  $B^2(\beta, x_-, t_+)$ ,  $B^2(\beta, y_+, z_+)$  are now vertical planks, based on the  $[x_-, t_+]$ ,  $[y_+, z_+]$  and going to  $\infty^{\text{ty}}$ .

When the “talk at  $\infty^{\text{ty}}$ ” where  $(\beta)$  lives, gets contracted to a point, then we get our Figure (B) and its likes. This is, of course, equivalent to the coning description given earlier.

**Lemma 11.2.** 1) *Using the crossing-freedom we can make that all the transversal contacts  $\text{body } \mathcal{B}_i^2 \cap \text{body } \mathcal{B}_j^2$  are RIBBONS. Typically, one starts from  $B^2(\beta, z_+, y_+) \cap B^2(\beta, x_-, t_+)$  in Figure 24-(B), and then one continues along  $P \times [r, b]$ , from  $r$  to  $b$ , like in Figure 25. Here, at the level of the  $b$  of  $2X_0^2$  we are far from  $\Delta^2 \times (\xi_0 = -1)$  and we can make full use of the crossing freedom.*

2) *From (90) it follows that all the contacts  $\text{body } \mathcal{B}_i^2 \cap \text{body } d_k^2$  are RIBBONS like in Figure 21-(A), coming always with  $d_{i_1}^2$  (long branch)  $\Rightarrow \mathcal{B}_i^2$  and  $d_{i_2}^2$  (short branch)  $\Rightarrow d_k^2$ . RIBBONS like in 1) above and like in the present 2) are not isolated but they, generally speaking, form highly connected and non-simply connected **nets**. This means the following FACT:*

(92.1) When, for the map

$$Z^2 \equiv \sum_1^M \text{body } \mathcal{B}_i^2 \cup \sum_1^N \text{body } d_k^2 \xrightarrow{G} \partial N^4(\Gamma_1(\infty)),$$

and for some transversal contact (= puncture)

$$p \in \{ \text{Curve } C \text{ from } \{\text{link}\} (9) \mid X^2[\text{new}] \} \cap G(\text{body } \mathcal{B}^2 \cup \text{body } d^2),$$

we look for an arc  $\lambda$ , embedded inside the source of  $G$ , connecting  $p$  to the  $\partial \mathcal{B}^2$  and free of accidents, then we may be OBSTRUCTED by contacts  $\lambda \cap \text{RIBBON}$  (and never by CLASPS, see here Figure 22).

3) Let  $e = [P_1, P_2]$  be an edge NOT living at  $\xi_0 = -1$ . In Figure 7 we see attached to  $e$  a rectangle (possibly with only a boundary collar alive). At the level of  $2X_0^2$ , we can always find inside this “rectangle” a much thinner one,  $e \times [r, \beta]$  (not affected by the deletion in Figures 7-(II, III), BUT non existent (wiped out), with the exception of  $(e \times r) \cup \{P_1, P_2\} \times [r, \beta]$  in Figure 7.bis, at  $\xi_0 = -1$ . Changing now the topic, we move next to the source  $Z^2$  of  $G$ , see (91), and we assume that for some  $u \in (x, y, z, t, \xi_0)$ , in the  $B^3(-) \subset S^3(P_1)$ , in the corresponding Figure 9, we find a  $b^3(\pm u)$ . If  $P_2$  is a vertex adjacent to  $P_1$  in  $2X_0^2$ , along the  $u$ -axis, then inside  $B^3(-) \subset S^3(P_2)$  we have to find  $b^3(\mp u)$ .

Moreover, as a reflex of our thin rectangle above, when we are OUTSIDE of  $\xi_0 = -1$ , then for every such  $u$ , we find at the level of  $Z^2$ , source of  $G$  (91), living inside  $\partial N^4(\Gamma_1(\infty))$ , a thin rectangle  $[\beta(P_1), \beta(P_2); u_{\pm}(P_1), u_{\mp}(P_2)]$ , going along it. (This could be the shaded rectangle in Figure 22, for instance.

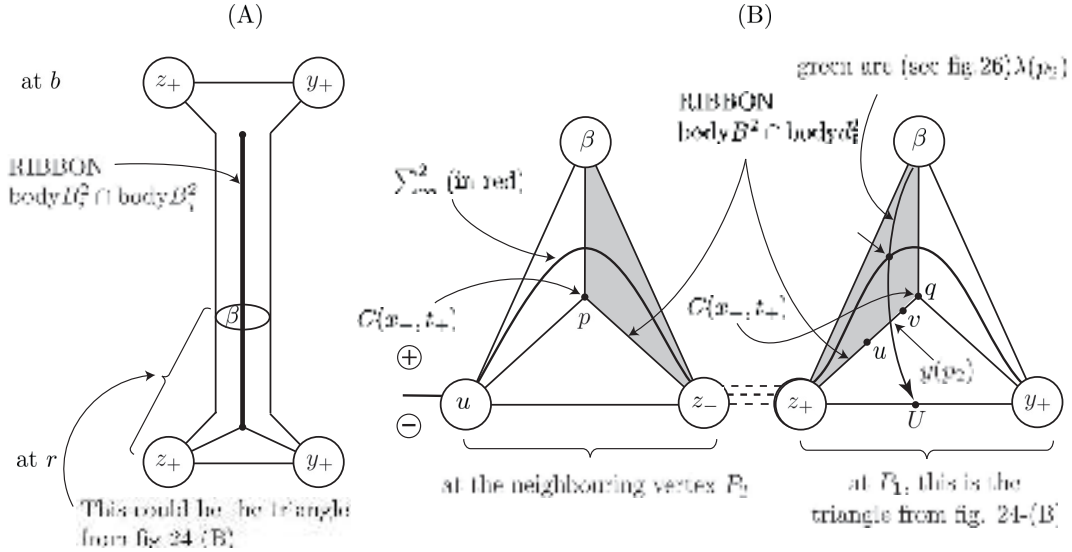


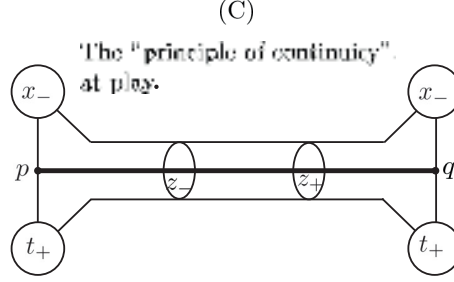
Figure 25.

This figure explains the occurrence of the two types of RIBBONS for  $Z^2 \xrightarrow{G} \partial N^4(\Gamma_1(\infty))$  (91), namely RIBBONS  $B^2 \cap B^2$  in (A) and RIBBONS  $B^2 \cap d^2$  in (B).

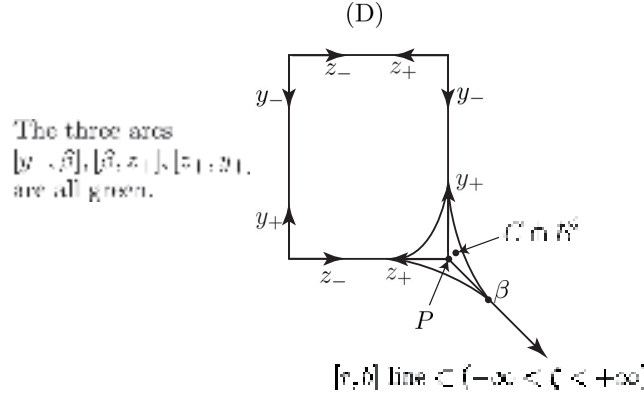
In Figure (B) the two shaded triangles are adjacent to the rectangle  $[\beta(P_1), \beta(P_2); z_+(P_1), z_-(P_2)]$ , which communicates with the rest of  $\delta^2 \mid X^2 \times r$  via them. This **rectangle** in question, is like in 3) in the Lemma 11.2. Similarly, in Figure 22 we have shaded the rectangle

$$[\beta(P \times (\xi_0 = -1)), \beta(P \times (\xi_0 = 0)); +\xi_0 \text{ (at } \xi_0 = -1), -\xi_0 \text{ (at } \xi_0 = 0)].$$

In (B), the L.H.S. triangle is the natural continuation of the one from the R.H.S., i.e. the one from Figure 24-(B). There is a “**principle of continuity**”, meaning that in moving from  $P_1$  to  $P_2$  the body  $d^2$ , body  $B^2$  has to use the common  $2^d$  walls in  $2X_0^2$ , valid both for  $P_1$  and for  $P_2$ . Figure (C) should illustrate this, and we see there the RIBBON body  $B^2 \cup \text{body } d_k^2$ , in all its glory.



We see here a piece of body  $d_k^2$ , containing the RIBBON  $[p, q] = \text{body } d_k^2 \cap \text{body } \mathcal{B}_i^2$ . In  $\partial N^4(\Gamma_1(\infty))$ , where this figure lives, the branch body  $\mathcal{B}_i^2$  cuts transversally the branch body  $d_k^2$ , through  $[p, q]$ . Finally, Figure (D) should suggest how a triangle like  $B^2(\beta, y_+, z_+)$  gets generated. In (D) this triangle is represented in green lines.



End of Figure 25.

The Lemma following next is a “worst case scenario”, in the sense that we show how to handle all the accident which, with what was done so far in the present paper are a priori possible, not excluded by anything we have said explicitly, yet. It will turn out that the real-life situation will be slightly better than the worst case scenario below.

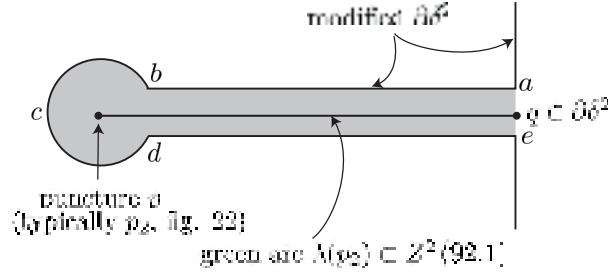
**Lemma 11.3. (Destroying the accidents.)**

1) *There are two procedures for destroying the accidents of (80) which we shall use:*

- I) *The push over the  $\{\text{extended cocore } (z)\}^\wedge$  (and when this cocore is not there, our I) certainly cannot be used, for instance when  $z \in \Delta^2 \times (\xi_0 = -1)$ ). Otherwise it is always there. Like in Figure 20-(B), this movement destroys both  $z$  and the adjacent double point  $x$ . But we **cannot** use I) for all the  $z$  which come with a cocore.*
- II) *The previous description of I) was at the level of  $N_1^4(2X_0^2)^\wedge$ , and we revert now to the  $Z^2 \xrightarrow{G} \partial N^4(\Gamma_1(\infty))$  from (92.1) and consider a puncture*

$$p \in Z^2 \cap \{a \text{ curve } C \in \{\text{link}\}\},$$

*which in  $4^d$  generates an accident  $z \in J\delta^2 \cap 2X_0^2 \subset N_1^4(2X_0^2)^\wedge$ . If for  $p$  we can find an embedded arc  $\lambda \subset Z^2$ , joining  $\lambda$  to a point  $q \in \partial \mathcal{B}^2 \approx \partial \delta^2$ , then we can proceed as in Figure 26, changing  $Z^2 \xrightarrow{G} \partial N^4(\Gamma_1(\infty))$ , and accordingly the (80) too, so as to destroy  $z$ . This modifies, of course  $(\delta^2, \partial \delta^2)$ .*



**Figure 26.**

Pushing  $\partial\delta^2$  along a green arc  $\lambda$ . In  $4^d$ , this destroys an ACCIDENT  $z$ . Then, the closed loop going through  $c, b, d$ , around  $p$ , when lifted off  $\partial N^4(2\Gamma(\infty))$ , by a multiplication with  $+\varepsilon$  (i.e. loop  $\Rightarrow$  loop  $x(+\varepsilon)$ ), is linked with  $2X_0^2 \subset N_1^4(2X_0^2)$ .

[With more detail, here is what goes on. Let  $A^2 \supset \lambda$ , with the  $A^2 \subset Z^2$ , being a connected piece, **without** internal accidents, and lifting  $Z^2 \rightarrow \partial N^4(\Gamma_1(\infty))$  into  $4^d$  means, for  $A^2$ , an embedded tubular neighbourhood  $A^2 \times [0, \varepsilon] = \varepsilon(A^2) > 0] \subset N_1^4(2X_0^2)^\wedge$ , with  $A^2 \times \{0\} = \{\text{our } A\}$ , with the rest  $A^2 \times (0, \varepsilon]$  in  $N_1^4(2X_0^2)^\wedge - N^4(2X_0^2)^\wedge$ , and with  $A^2\{\varepsilon\} \subset JZ^2 \subset J\delta^2$ . For our green arc  $\lambda$  we consider the shaded neighbourhood  $\nu^2(\lambda) \subset A^2$ , from Figure 26, and its  $\nu^2(\lambda) \times [0, \varepsilon]$ . (The  $\varepsilon$  may be happily variable here, this does not matter.) The modified  $J\delta^2$ , with the accident  $z$  destroyed, is gotten by deleting  $\{\text{the contribution } \nu^2(\lambda) \times [0, \varepsilon] \subset A^2 \times [0, \varepsilon]\} - \{\text{its boundary piece, i.e. [the arc } (a, b, c, d, e), \text{ Figure 26}] \times [0, \varepsilon]\}$ . Of course, for this construction to be possible, it is necessary that on its road from  $q \in \partial\delta^2$  to the puncture  $p$ , the green arc  $\lambda(p_2)$  should meet **NO OBSTRUCTION**; and this issue will be discussed at length. Also we may assume that  $\partial Z^2 \supset \partial\delta^2$ . With this construction comes a modification of  $\partial\delta^2$ . The very short arc

$$q \in [a, e] \subset \partial\delta^2$$

is to be replaced by the long  $[a, b, c, d, e]$ . We will denote this change, for  $\delta^2 = \delta_i^2$  by

$$c(b_i) \Rightarrow c(b_i)(\lambda).]$$

**Important Remark.** When we consider

$$\lambda(p_2) \subset GZ^2 \subset \partial N^4(\Gamma_1(\infty)),$$

typically the  $N^4(\Gamma_1(\infty))$  can see inclusions coming from the  $\{h\}'s \subset \text{LAVA}$

$$(B^3 \times [-\varepsilon, \varepsilon], S^2 \times [-\varepsilon, \varepsilon]) \subset (N^4(\Gamma_1(\infty)), \partial N^4(\Gamma_1(\infty))),$$

coming with contacts  $\lambda(p_2) \cap S^2 \times [-\varepsilon, \varepsilon] \neq \emptyset$  not touching to the  $C$ 's of the  $D^2(C) \subset \text{LAVA}$ . Let us say that we may a priori see contacts

$$(*) \quad \lambda(p_2) \cap \{\text{extended cocore } z\} \neq \emptyset.$$

It will turn out that the procedure from Figure 35.4 in the next section V will demolish all the contacts  $\{\lambda(p_2) \cap d_k^2\} \cap \{\text{cocores } B \text{ or } R\}$ .

On the other hand, the totality of the contacts

$$\{\lambda(p_2) \text{ or } \lambda(q)\} \cap \{h \in R\}$$

which will turn out to mean, always  $h \in R - B$ , will be, all of them, when dangerous for us, part of the parasitical terms in (139), and they will be all taken care of by the CHANGE OF COLOUR, at the end of section VII. So, contacts  $\{\lambda(p_2) \text{ or } \lambda(q)\} \cap R$  should not worry us, but only  $\{\lambda(p_2) \text{ or } \lambda(q)\} \cap B$ , and they will be all taken care of too. End of the Important Remark.

2) Because of the lack of extended cocores, the transversal contacts

$$z_2 \in Jd^2(+\xi_0, \text{space-time}, \text{space-time}) \pitchfork (\Delta^2 \times (\xi_0 = -1))$$

which correspond to the punctures  $p_2$  in Figure 22 (CLASP) obviously need treatment II. To the same  $p_2$ 's are associated RIBBONS

$$d^2(+\xi_0, \text{space-time}, \text{space-time}) \pitchfork d^2(+\xi_0, \text{space-time}, \text{space-time}),$$

like the  $[p_2, z_+], [p_2, y_+], \dots$  Figure 22. The only punctures at  $\xi_0 = -1$  are the  $p_2$ 's coming with  $p_2 \in \text{body}(d^2 \mid \Delta^2 \times (\xi_0 = -1)) \pitchfork C(\text{curve of } \Delta^2 \times (\xi_0 = -1)) \subset \partial N^2(\Gamma_1(\infty))$ .

The Figure 22 suggests that the clasps  $\text{body}d^2 \pitchfork \text{body}d^2$  are not an obstruction for  $\lambda(p_2)$ . For further purpose notice also that

$$(\lambda(p_2) \mid [P \times [0 \geq \xi_0 \geq -1]]) \cap (B_0 \cup R_0) = \emptyset.$$

The treatment of the RIBBONS at  $\xi_0 = -1$  is suggested in Figure 28-bis, while the treatment of the RIBBONS at  $\xi_0 = 0$  is suggested in the Figure 29, WHEN FIGURE 22 is to be used. Notice how this fits with the LHS of the Figure 23. It is important here that at  $P \times [0 \geq \xi_0 \geq -1]$  we have a CLASP and not a RIBBON.

3) When we join the  $p_2$  above with the corresponding  $c(b_i) = \partial\delta_i^2$ , by an embedded **long green arc**  $\lambda(p_2) \subset Z^2$  (91), and here it should be understood that the various  $\lambda(p_2)$ 's are two-by-two disjointed, then all the contacts (which are certainly **in the way** for the process in Figure 26)

$$y \in \lambda(p_2) \pitchfork \{ACCIDENTS \text{ of } Z^2 \xrightarrow{G} \partial N^4(\Gamma_1(\infty))\}$$

are some intersections with RIBBONS  $\text{body}\mathcal{B}^2 \pitchfork \text{body}d^2$  (case  $\bullet$ ) below) or RIBBONS  $\text{body}d^2 \pitchfork \text{body}d^2$  localized at  $\xi_0 = 0$  (case  $\bullet\bullet\bullet$ ) below). The  $y = y(p_2)$  occurring here is like in Figure 27 and RIBBONS  $\text{body}\mathcal{B}^2 \pitchfork \text{body}d^2$  containing such an obstruction are called **very special**. Their treatment is explained in Figure 28. For the obstructing RIBBONS at  $\xi_0 = 0$ , the treatment is explained in {Figure 29 with the change  $\mathcal{B}^2 \Rightarrow d^2$ }.

Now, in the same connected component of the union of RIBBONS as the very special RIBBONS, there an other ribbons  $\text{body}\mathcal{B}^2 \pitchfork \text{body}d_k^2$ , which we will call **special** RIBBONS. It may happen that a very special RIBBON contains several  $y(p_2)$ 's or that very special RIBBONS are adjacent to each other. In order to simplify the exposition, we will assume that none of these things happen, but our procedures extend easily to these more general cases.

4) When a CLASP is lifted to  $4^d$  then this **has** to come with one double point  $x \in JM^2(J)$ , which can be placed at either end of the clasp. When a RIBBON is lifted to  $4^d$ , then there is a **FREE CHOICE** which can be made independently for each individual RIBBON: either we create two double points  $x_1, x_2 \in JM^2(J)$  OR none.

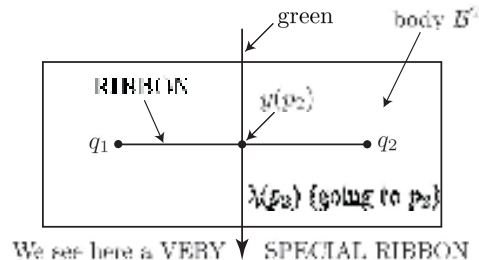


Figure 27.



A contact  $y(p_2) \in \{\text{green arc } \lambda(p_2) \cap \{\text{very special RIBBON } [q_1 q_2]\}, \text{ obstructing the } \lambda(p_2), \text{ like the } \lambda(p_2) \text{ in Figure 22.}$

Here is how this margin of freedom will be made use of. There is, to begin with, the category of RIBBONS called **generic**, for which the (imposed) choice is **zero double points**, and these ribbons are the following ones:

•) The **very special RIBBONS** and the **special RIBBONS** coming with them, all of type  $\text{body } \mathcal{B}^2 \cap \text{body } d^2$ . Also, the RIBBONS  $\text{body } \mathcal{B}^2 \cap \text{body } \mathcal{B}^2$  from Figure 25-(A), when they are in same connected component with them.

••) The RIBBONS  $\text{body } d^2 \cap \text{body } d^2$  localized at  $\xi_0 = -1$ , always having endpoints  $p_2$ . Since  $p_2$  comes with the treatment of the long  $\lambda(p_2)$ (green), contrary to the case •), **SHORT GREEN ARCS**, like in the Figure 28, are useless here. Then RIBBONS occur in the RHS of the Figure 23 and again in Figure 28-bis.

[These RIBBONS are like in Figure 25-(B), with the following changes:

- i) The  $\beta$  becomes  $+\xi_0$ .
- ii) The triangles one sees in Figure 25-(B) become now the  $d^2(+\xi_0, \text{space-time}, \text{space-time})$ ,  $d^2(+\xi_0, \text{space-time}, \text{space-time})$ .
- iii) The  $p, q$  become both  $p_2$ 's.
- iv) The curves  $C$  from Figure 25-(B) are now  $\Gamma_j$ 's, since they live at  $\xi_0 = -1$ .

The important fact for the section V (CONFINEMENT) and also for the Lemma 12, is that both at  $P_1$  and at  $P_2$ , to which the two  $p_2$ 's pertain, the  $\Gamma_j$ 's which contain them are UP; the Figure 24-(A) shows this fact. Notice that, for a puncture  $p_2 \in d^2(+\xi_0, \text{space-time}, \text{space-time}) \cap "C"$  to be there, we have to have a crossing of curves at which our "C" should be UP.]

We move next to the other category of RIBBONS, called **exceptional**, for which our choice will be  $\#JM^2(J) = 2$ . These are:

•••) All the non-special RIBBONS  $\text{body } \mathcal{B}^2 \cap \text{body } d^2$ , as well as the RIBBONS  $\text{body } \mathcal{B}^2 \cap \text{body } \mathcal{B}^2$  in the same connected component.

••••) The RIBBONS  $\text{body } d^2 \cap \text{body } d^2$  connected with the puncture  $p_1$  at  $\xi_0 = 0$ , see Figures 22, 23. They occur in the LHS of Figure 23, and the paradigmatic figure for handling them is 29, for all the exceptional RIBBONS too.

In all this story, RIBBONS  $\text{body } \mathcal{B}^2 \cap \text{body } d^2$  from Figure 25-(B) (or 25-(C)) are always treated on the same footing. According to the case, they are treated like in Figure 28 OR in Figure 29. Anyway,  $\mathcal{B}^2$  is always the long branch and  $d^2$  the short one.

More comments will follow now, concerning the •) above. So look at the scenario from Figure 28, explained in 5) below.

5) What Figure 28 presents, is the scenario via which, in the context of our •), the long green arc  $\lambda(p_2)$  is freed at  $y(p_2)$  and hence allowed to continue towards the accident  $z_2$  attached to the transversal contact  $p_2 \in F \text{body } d_k^2 \cap \Gamma_j \times (\xi_0 = -1)$ , in the Figure 23. The procedure (II) (Figure 26) is being used here at  $p_2$ . The basic idea is here the following: Once a RIBBON is generic and  $(\mathcal{B}^2, d^2)$  are involved, there are **no**  $x_1, x_2 \in JM^2(J)$ , and we have the inequality

$$(92.2) \quad \varepsilon(\text{branch}, \mathcal{B}^2) < \varepsilon(\text{branch } d^2).$$

This allows us to proceed like in Figure 28.

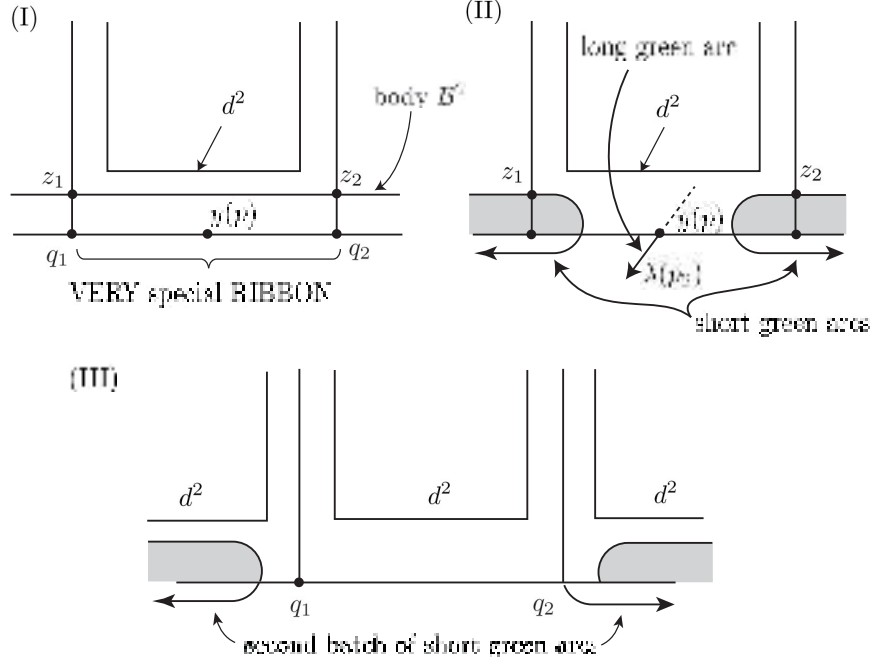


Figure 28.

Liberating the main green arcs  $\lambda(p_2)$ . From  $q_1, q_2$  on, we continue to demolish the accidents in the whole connected component of special RIBBONS, via short green arcs. ONLY procedure II is present in this figure. The common numerals (I), (II), (III) correspond to the time ordering of the successive steps.

6) Figure 28 suggests how the long green arc  $\lambda(p_2)$  is able to get to the accident  $z_2 \in Jd^2 \cap \Delta^2 \times (\xi_0 = -1)$  occurring at the puncture  $p_2$  from the Figure 22. **Afterwards**, for the accidents  $z_i \in J\delta^2 \cap 2X_0^2$  occurring at the punctures  $q_i \in \text{body}\mathcal{B}^2$  along the whole connected component of very special and special RIBBONS  $\text{body}\mathcal{B}^2 \cap \text{body}d^2$  from Figure 28, occurring up-stream from  $p_2$ , one uses the short green arcs  $\lambda(q)$  suggested in Figure 28. The point here (Figure 28) is that, with

$$\varepsilon(\nu^2(\lambda)) = \varepsilon(\text{branch}\mathcal{B}) < \varepsilon(\text{branch}d^2),$$

the high  $Jd^2$  is no longer in the way for  $\lambda(p_2)$  which can happily proceed under it, inside the branch  $\mathcal{B}^2$ . When we get to  $\xi_0 = -1$  then we similarly have

$$\varepsilon(\nu^2(\lambda)) = \varepsilon(\text{branch}d^2(+\xi_0, \dots), \text{carrying the puncture } p_2) < \varepsilon(\text{the dual } d^2\text{-branch}),$$

and this allows us to proceed like in Figure 28-bis, which takes care of  $p_2$  (and  $z_2$ ).

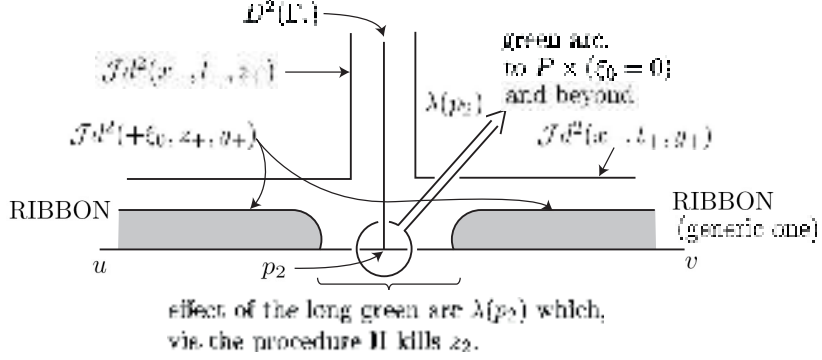


Figure 28-bis.

We are here at  $P \times (\xi_0 = -1)$  and the  $\lambda(p_2)$  lives in a plane which is transversal to our figure, cutting through  $[u, v]$ . See here also the Figures 22 and 23.

All this takes care of the situation  $\bullet$ ) and, as already said earlier, the  $\bullet\bullet$ ) is to be taken care of by Figure 28-bis, which does **not** need short green arcs.

At this point, the only ACCIDENTS still alive are the transversal contacts  $z \in J\delta^2 \cap 2X_0^2$  coupled with double points  $x \in JM^2(J)$  (with  $J$  like in (80)) coming with the exceptional ribbons from  $\bullet\bullet\bullet$ ) +  $\bullet\bullet\bullet\bullet$ ) at the end of 4) above.

[The ribbons  $\text{body}\mathcal{B}^2 \cap \text{body}\mathcal{B}^2$ , like in the Figure 25-(A) are either generic, i.e. connected with the special and very special RIBBONS OR exceptional. The generic ones will get the treatment of short given arcs from Figure 28, while the exceptional ones will be treated in Figure 29.]

When it comes to the exceptional RIBBONS we make the choice opposite to the one made in connection with Figure 28, namely now

$$\varepsilon(\text{short branch}) < \varepsilon(\text{long branch})$$

coming with two double points  $x \in JM^2(J)$  for each exceptional RIBBON.

We have now Figure 29, and we apply the treatment (I) which simultaneously kills the  $z$  and the  $x$ 's. Notice that for the global killing of our ACCIDENTS, both procedures (I) and (II) are needed.

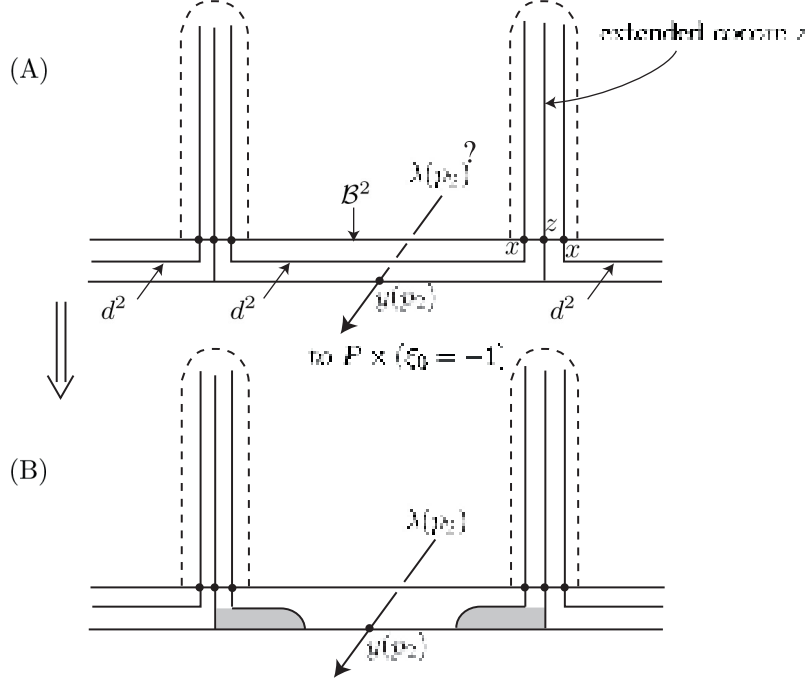
In the context of Figure 29 for  $\bullet\bullet\bullet\bullet$ ) we may have possible obstructions

$$\lambda(p_2) \cap \{\text{RIBBONS} \dots\}$$

which are then treated like in Figure 29-(B), without any short arcs.

With this, the proof of our Lemma 11.3 is finished, but NOT YET the proof of Theorem 11. We have gotten rid of all the ACCIDENTS but the diagonalization condition (82.1), which we will call THE LITTLE BLUE DIAGONALIZATION is not yet realized. That will be done in the next section V.

We end this section with some last COMMENTS.



**Figure 29.**

Illustration for the procedure I for killing ACCIDENTS (transversal contacts  $z$  and double points  $x$ , simultaneously). The  $\lambda(p_2)$  with the potential obstruction  $y(p_2)$  may be present for  $\bullet\bullet\bullet\bullet$ . [This same figure applies at  $\xi_0 = 0$ , with  $B^2 \Rightarrow d^2$ , IN THE CASE WHEN FIGURE 22 IS USED (CLASP along  $[0 \geq \xi_0 \geq -1]$ .) It is **only** then that  $y(p_2)$  may be present.]

We go back now to our family

$$(93) \quad \sum_1^M b_i = B_1 \cap (\Gamma(1) \times (\xi_0 = -1)) = B_0 \cap (\Gamma(1) \times (\xi_0 = -1)).$$

Notice here that each edge  $e \subset \Gamma(1) \times (\xi_0 = -1)$  contains its  $b_i \in \sum_1^M b_i$ . On the other hand, since  $\Gamma(1) \times (\xi_0 = -1) = \sum_1^n R_i \times (\xi_0 = -1)$  is a tree, we have  $M \gg n$ , i.e. our  $\Delta^2 \times (\xi_0 = -1)$  is violently disbalanced, as far as the RED/BLUE balance is concerned.  $\square$

Notice that the destruction of accidents, via I) changes the  $J\delta_i^2$  modulo the boundary, which stays unchanged while II) changes the boundary too. So, we will distinguish between the **original**  $C(b_i)$ 's and the  $C(b_i)(\lambda)$ 's  $\equiv \{c(b_i) \text{ MODIFIED by the effect of the green arcs } \lambda(p_2) \text{ (and the short green arcs)}\}$ . It is the

$$(93.1) \quad \left( \sum_1^M \delta_i^2, \sum_1^M \partial \delta_i^2 = C_i(b)(\lambda) \right) \xrightarrow{J} (\partial N^4(2X_0^2)^\wedge \times [0, 1], \partial N^4(2X_0^2)^\wedge \times \{0\}),$$

which is ACCIDENT-free.

Now, to each  $b \in B_0 = \{\text{the } b \in X^2[\text{new}] \subset 2X^2 \text{ which live on the } X^2 \times r \text{ side}\}$ , corresponds on the  $b$ -side, and see here the Figures 7-II, III and 7-bis, a  $b \times b \in B_1$ , on the  $X^2 \times b \approx X_b^2$  side.

(94) In the geometric intersection matrix, for  $1 \leq i \leq M$  we find, BEFORE Lemma 11.3 has been applied, for the original  $C(b_i)$ ,

$$C(b_i) \cdot b_i = 1, C(b_i) \cdot (b_i \times b_i) = 1 \text{ and NOTHING else.}$$

When we move from  $C(b_i)$  to  $C(b_i)(\lambda)$ , then we find

(94.1)  $C(b_i)(\lambda) \cdot b_i = 1$  AND ALSO OFF-DIAGONAL TERMS, both  $C(b_i)(\lambda) \cdot \{\text{the } b_i \times b_i\} = 1$  and  $C(b_i)(\lambda) \cdot \{\text{various } b \in B_0 \text{ touched by the green arcs, long or short}\} \neq 0$ .

The off-diagonal terms  $C(b_i)(\lambda) \cdot B$  above, are called PARASITICAL. Since  $\lambda(p_2)$  never goes along an edge  $e \subset \Gamma(1) \times (\xi_0 = -1)$  and since there are no short green arcs at  $\xi_0 = -1$ , the  $\sum_{i=1}^M b_i$  are never parasitical.

(94.2) Here is a definition which will be useful. Let  $b \in B_0$  and let  $b = b_0 \rightarrow b_1 \rightarrow \dots \rightarrow b_k$  be its normal blue trajectory, defined like in (77.1). This comes with successive  $B^2(b_i)$ 's. We will say that  $b$  is **trivial**, if for **none** of them  $B^2(b_i)$  we have inclusions  $D^2(\gamma^0) \subset B^2(b_i)$ . If, moreover  $k = 0$ , i.e. if  $b$  has no outcoming arrows, then we say that it is **very trivial**.

## 5 Confinement and the little blue diagonalization

We start by making the following CLAIM, concerning the edges  $e(r) \subset \{\text{curve } c(r)\} \subset \{\text{link}\}$ , displayed in the Figures 7-(I and IV).

**Claim (95).** Using our margin of freedom for crossings which do not concern  $\Delta^2 \times (\xi_0 = -1)$ , we can ask that for all the Figures 9 (and/or 24) concerned, and at all its corners, every  $e(r) \subset c(r)$  (Figure 7) should be UP.

**Proof.** We will consider afterwards the case when  $e(r) = P \times [0 \geq \xi_0 \geq -1]$ . So we look now at  $e(r)$  going from  $P_1$  to  $P_2$  along an axis  $u \in (x, y, z, t)$ .

Let us say that  $e(r)$  goes from  $b^3(u_{\pm})$  at  $P_1$  to  $b^3(u_{\mp})$  at  $P_2$ . The corners of  $c(r)$  at  $e(r)$  are then  $(u_{\pm}, \beta)$  at  $P_1$  and  $(u_{\mp}, \beta)$  at  $P_2$ . Figure 24 tells us that all the arcs  $[b^3(u_{\pm}), b^3(\beta)]$  are UP at all their crossings, which is what our claim says, concerning them.

Similarly, for the case  $e(r) = P \times [0 \geq \xi_0 \geq -1]$  the arcs  $[b^3(\pm \xi_0), b^3(\beta)]$  are UP too. End of Proof.

Once we have the CLAIM above, without any obstruction the  $c(r)$ 's can be pushed into  $\partial N_+^4(2\Gamma(\infty))$ .

Next, we want to describe a class of **admissible subdivisions** of  $2X_0^2$  which should guarantee, at the BLUE level, the conservation of the following basic feature, part of at the level  $X^2[\text{new}]$  (and of course at  $2X^2$  too)

$$\boxed{\{\text{curves } \Gamma \text{ at } \xi_0 = -1\} \subset \{\text{curves } \gamma^1\}} .$$

Remember that, for general subdivisions, the RULES OF THE GAME are

$$D^2(\gamma^0) \implies \{\text{one small } D^2(\gamma^0)\} + \{\text{many small } D^2(C)\text{'s}\} \text{ (RED)}$$

$$D^2(\gamma^1) \implies \{\text{one small } D^2(\gamma^1)\} + \{\text{many small } D^2(\eta)\text{'s}\} \text{ (BLUE)}.$$

That is why we need now special, "admissible subdivisions". Our admissible subdivision is presented in Figure 30 and, at  $\xi_0 = -1$ , we see there a lot of  $B(\text{new})$ 's only one of which is  $B(\text{new}) \cap R$ . This way, each edge of  $\Gamma(1) \times (\xi_0 = -1)$  continues to have one  $B$  and  $\Gamma(1) \times (\xi_0 = -1) - R$  continues to be a tree.

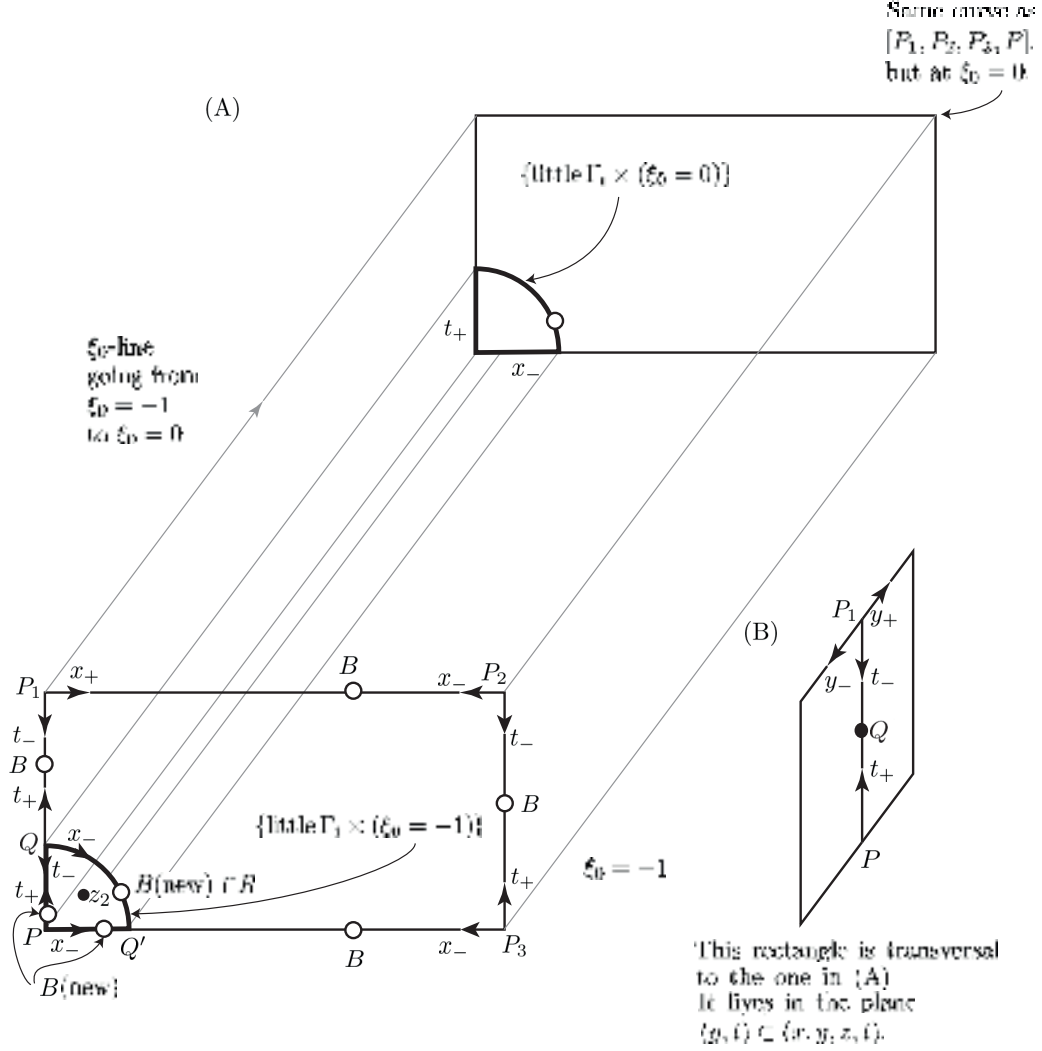


Figure 30.

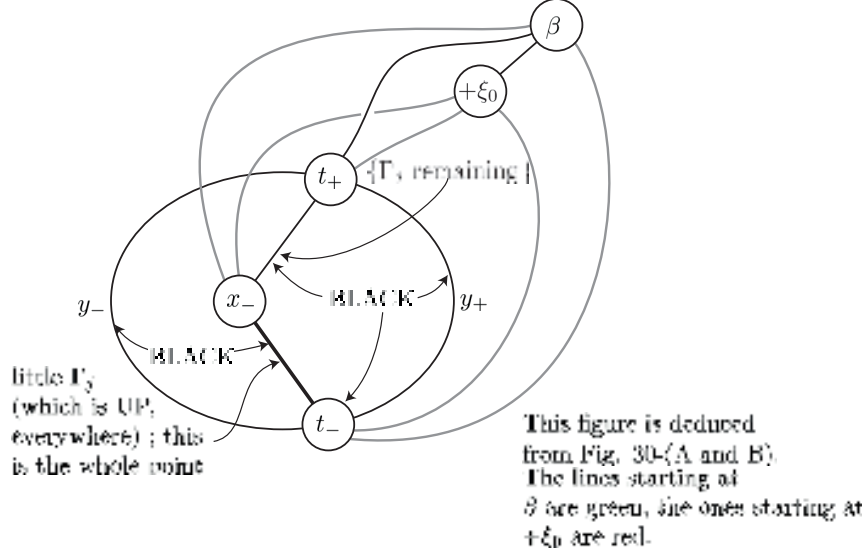
Here  $[P, P_1, P_2, P_3]$  is a generic curve  $\Gamma_j \subset \Delta^2 \times (\xi_0 = -1)$  with, let us say  $P$  the vertex of Figure 22, part of a Figure 9, as displayed in Figure 24. Just like, at  $\xi_0 = -1$ , the  $\Gamma_j \times (\xi_0 = -1)$  breaks (BLUE-wise) into many little  $\gamma^1$ 's, at  $\xi_0 = 0$ , the  $\Gamma_j \times (\xi_0 = 0)$  also breaks (RED-wise), in many little  $\gamma^0$ 's (and, of course, BLUE-wise in  $\gamma^1$ 's too).

**Explanations concerning Figure 30.** The  $z_2$ , placed close to  $P$  is an accident  $z_2 \in J\delta^2 \cap D^2(\Gamma_j)$  (= square  $[P, P_1, P_2, P_3]$ ). The  $\Gamma_j$  is UP at  $P$  (see Figures 22 and 24-A), but certainly not UP everywhere, at all vertices, at least there is no reason for that. We subdivide  $[P_1, P_2, P_3, P] \subset (\xi_0 = -1)$  like suggested, BUT then the whole  $[P_1, P_2, P_3, P] \times [1 \leq \xi_0 \leq 0]$  too. It is easy to see that this subdivision is now admissible, in the sense defined above. We use this subdivision at every corner  $P$  carrying an accident, i.e. a  $(p_1, z_2)$ .

This also breaks  $\Gamma_j$  into

$$\Gamma_j = \{\text{the **little** } \Gamma_j, \text{ i.e. the round triangle } [P, Q, Q'], \text{ keeping the accident } z_2\} + \\ + \{\text{the big } \Gamma_j \text{ remaining, which is free of accidents}\}.$$

Figure (B) suggests the environment of  $[P_1, P]$  in  $\Delta^2 \times (\xi_0 = -1) \subset 2X_0^2$ . Also now  $z_2 \in J\delta^2 \cap D^2(\text{little } \Gamma_j)$  and little  $\Gamma_j$  is UP at **all** its corners. This is the aim of the whole thing Figures 24 + 30 show that little  $\Gamma_j$  is UP at  $P$  and Figure 31 shows that it is UP at  $Q$  too. For  $Q'$  it is the same.



**Figure 31.**

This is the Figure 9 (complete with  $+\xi_0$  and  $\beta$ ) for the  $Q$ , the new vertex introduced in Figure 30. Here only the BLACK arcs are part of curves  $\Gamma_j \subset \Delta^2 \times (\xi_0 = -1)$  and at the crossings in which they take part, in this figure, they are always DOWN. The RED and GREEN curves are not part of  $\Delta^2 \times (\xi_0 = -1)$  and, with these things, this figure cannot create ACCIDENTS involving  $\Delta^2 \times (\xi_0 = -1)$ . There is a similar figure at  $Q'$  (see Figure 30-(A)). For the colours see the legend of Figure 24.

Our subdivision introduces an additional  $B(\text{new})$ 's, one of which is  $B \cap R$  which can all be treated like the  $b_i$ , in the context of Figure 19.1. Since  $\Gamma(1)$  increases, a new  $R = R_{n+1}$  has to be added to the already existing  $\sum_1^M R_i$ . This is, of course, our  $B(\text{new}) \in R \cap B$ , Figure 30-(A). Remember at this point that the only function of the set  $\Sigma R_i$  is to render the  $\Gamma(1) - \Sigma R_i$  be a tree. It will essentially disappear from our picture.

The construction above extends easily to the accidents carried by  $D^2(C)$ 's, outside  $\xi_0 = -1$  and which also need green arcs. We need now just a normal subdivision, without worrying about the admissibility condition, which is certainly not required outside of  $\Delta^2 \times [0 \geq \xi_0 \geq -1]$ . With this we will isolate any transversal contact  $z \in J\delta^2 \cap D^2(C)$ , proceeding like in the lower part of Figure 30, and perform, for each  $C$ , the change (in particular done for  $C = \Gamma_j \times (\xi_0 = 0)$ )

$$C \implies \{\text{the } \textit{little} C\} + \{\text{the } C \text{ remaining}\}.$$

**Important Remark.** The accident coming with the  $D^2(\Gamma_j) \times (\xi_0 = 0)$  in Figure 30 certainly does not need a green arc, since we apply for it the push over the  $\{\text{extended cocore}\}^\wedge$ . Nevertheless, for reasons to become clear later, we still apply to it the procedure from Figure 30. Here  $\{\text{little } C\}$  is UP at all its corners and retains the accident  $z$ .

Concerning now the SPLITTING (27.1), with things as they stood at the end of the preceeding section IV, we had the normal CONFINEMENT conditions, for the  $\{\text{link}\}$  (= the internal curves, attaching zones of 2-handles if  $2X^2$ )

$$(96) \quad \sum_1^\infty C_i(\text{of } X_0^2[\text{new}]) + \sum_1^{\bar{n}} \Gamma_j(\subset (\xi_0 = -1)) + \{\text{extended } \gamma^0\text{'s (18.1)}\} \subset \partial N_-^4(\Gamma_1(\infty)) \text{ AND } \sum_1^\infty \eta_i \times b + \sum c(r) \text{ (see claim (95))} \subset \partial N_+^4(2\Gamma(\infty)) \supset \sum_1^M c(b_i)(= \partial \delta_i^2).$$

Now, just like in the context of the CLAIM (95), for  $b_{i \leq M}$ , the  $c(b_i)$  is UP at all its corners (since in Figure 24 all the arcs  $[\beta, \text{space-time}]$  are UP) and so we also have  $\sum_1^M c(b_i) \subset \partial N_+^4(2\Gamma(\infty))$ . By contrast, for  $\sum_1^M c(b_i)(\lambda)$  this is not clear at all, and there are two issues to be settled for the green arcs, in their totality, the long green arc  $\lambda(p_2)$  and the short green arcs  $\lambda(q)$ :

**Issue A)** We want the confinement condition

$$\{\text{green arcs } \lambda\} \subset \partial N_+^4(2\Gamma(\infty)), \text{ to be satisfied.}$$

This would allow us to add  $\sum_1^M c(b_i)(\lambda) \subset \partial N_+^4(2\Gamma(\infty))$  to (96).

**Issue B)** We also want to have the following condition satisfied

$$\{\text{green arcs } \lambda\} \cdot B \subset \{\text{trivial } B\text{'s (see (94.2))}\}.$$

These will be essential for getting the (82.1) and hence clinch the proof of the Theorem 11. But notice, that with what we have already done in this section, we can replace (96) with the real life FORCED CONFINEMENT CONDITIONS to be used from now on, until we can get to something even better:

$$(97) \quad \sum \{C_i \text{ remaining}\} + \sum \{\Gamma_j \text{ remaining}\} + \sum \gamma_k^0 \subset \partial N_-^4(\Gamma_1(\infty)), \text{ and then } \sum \{\text{little } C_i\} + \sum \{\text{little } \Gamma_j\} + \sum \eta_i \times b + \sum c(r) \subset \partial N_+^4(2\Gamma(\infty)) \supset \sum_1^M c(b_i).$$

[The  $\{\text{little } \dots\}$  which we want to push into  $\partial N_+^4(2\Gamma(\infty))$ , have to be UP at all their corners. And this condition **is** fulfilled since these  $\{\text{little } \dots\}$  either come from the  $D^2(\dots)$ 's carrying accidents needing green arcs or the  $D^2(\Gamma \times (\xi_0 = 0))$  from Figure 30. For this last one there is no accident but it is UP at the relevant corner, since the  $D^2(\Gamma \times (\xi_0 = -1))$  is, Figure 24-(A).

All the other accidents needing green arcs come from body  $\mathcal{B}^2$  and Figure 24-(A) tells us that the  $B^2(\beta, \dots)$ 's are UP at all their corners.]

And once Lemma 12 and also Theorem 13 will be with us, we will be able to improve the RHS of (97), by setting

$$\partial N_+^4(2\Gamma(\infty)) \supset \sum_1^M \eta_i(\text{green}).$$

[The more precise way in which the confinement condition  $\sum c(r) \subset \partial N_+^4(2\Gamma(\infty))$  should occur, is explained in the Figure 45, which matches well with Figure 32, which displays the confinement

$$\sum \{\text{little } C_i\} \subset \partial N_+^4(2\Gamma(\infty)).]$$



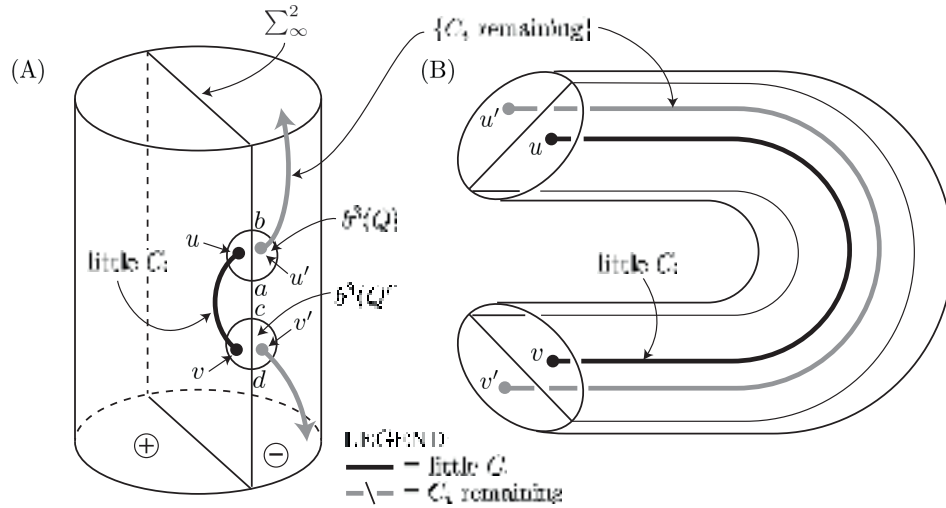
In a figure like 45 or 32, we are in  $\partial N^4(2\Gamma(\infty))$ , and even if only  $2\Gamma(\infty)$  is mentioned explicitly in our drawings (in Figure 45), this is so only for typographical reasons, since the  $2\Gamma(\infty)$  is buried deep inside  $N^4(2\Gamma(\infty))$ , far from its boundary. But the curves are there and, in Figure 45-(III), by  $c(b)$  we mean  $c(b_{i \leq M})$ . Forgetting, for the time being, about the issue B), here is how issue A) is taken care of.

**Lemma 12.** *Via a simple isotopic move of the map (see (92.1))*

$$Z^2 = \sum_1^M \text{body } \mathcal{B}_i^2 \cup \sum_1^N \text{body } d_k \xrightarrow{G} \partial N^4(2\Gamma(\infty)),$$

we can make that

$$(98) \quad \sum \{\text{green arcs } \lambda\} \subset \partial N_+^4(2\Gamma(\infty)).$$



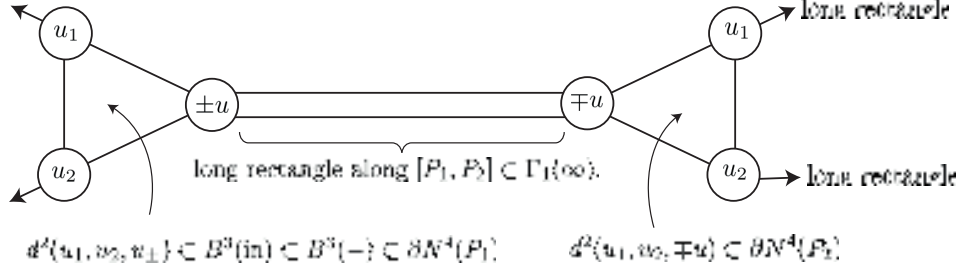
**Figure 32.**

The  $4^d$  geometry of  $\{\text{little } C_i\} \subset \partial N_+^4(2\Gamma(\infty))$ ; (and of course, this applies to  $\{\text{little } \Gamma_i\}$  too). We have a  $B^4(P)$  such that  $C_i$  occurs as an arc in  $\partial B^4(P)$ . In (A) we have suggested a  $B^3 \times [-\varepsilon, \varepsilon]$  cutting through  $(B^4(P), C_i)$  such that  $\partial B^3 \times [-\varepsilon, \varepsilon] \subset \partial B^4(P)$ , this is the visible part in Figure 32-(A) and it is far from any other  $C$  which may be in the way, between our  $C_i$  and  $\partial_+ N^4(2\Gamma(\infty))$ . The (B) shows the 1-handle necessary for the transformation

$$C_i \implies \{\text{little } C_i\} + \{C_i \text{ remaining}\}.$$

In order to see the  $\{\text{little } C_i\}$  close in, we have to put the (A) and (B) together.

**Proof of Lemma 12.** We have to look much closer at body  $d^2 \cup \text{body } B^2 \subset \partial N^4(\Gamma_1(\infty))$ , inside which the green arcs are, anyway, contained. Each  $d_k^2$ —body  $d_k^2$  is a disjoint collection of discs, while  $\partial d_k^2 \subset \partial \text{body } d_k^2$ .



**Figure 32.1.**

A piece of body  $d_k^2 = U(\text{triangles}) \cup U(\text{rectangles})$ , continuing along the four arrows. The four sides of the figure are contained in  $UC_i \subset \{\text{link}\}$ .

The disc with many holes, the body  $d_k^2$ , is body  $d_k^2 = \left( \bigcup \{ \text{individual triangles } d^2(u_1, u_2, u_3), \text{ where } u_i \in (\pm x, \pm y, \pm z, \pm t, \pm \xi_0), \text{ contained inside the } \partial N^4(P) \text{'s. They have disjoint interiors and they all occur inside the various embellished Figures 9} \} \right) \cup \sum \{ \text{long rectangles, contained inside the various } \partial N^4(\Gamma_1(\infty)) \mid \{ \text{edge } [P_1, P_2] \} \}$ , and they join two triangles  $d^2(u_1, u_2, u_3)$ , like in the Figure 32.1 below}.

In terms of the MODEL FOR  $N^4(\Gamma_1(\infty))$  given around formulae (27) to (27.5), for  $d^2(u_1, u_2, u_3) \subset B^3(-)$ , there is a vector field along  $d^2(u_1, u_2, u_3)$  normal to it and looking towards  $\Sigma_\infty^2$ . In terms of the Figure 24, which is the paradigmatical figure of type 9, drawn as a sort of link diagram on its  $S^2(P)$  (and/or  $S_\infty^2$ ), this is the direction looking towards the observer, hence, when one is at a crossing, then

$$\vec{v} = \{\text{direction DOWN} \rightarrow \text{UP}\}.$$

When we consider a figure like 32.1, the  $\vec{v}$  extends **continuously** throughout such a figure, hence continuously along the  $[P_1, P_2]$ . Here is how one has to understand what is going on in a complete Figure 32.1. We look again at our paradigmatical Figure 24, and let us say that we want to see what goes on along  $P \times [-1 \leq \xi_0 \leq 0]$ . Figure 24 displays the  $\partial N^4(P \times (\xi_0 = -1))$  and inside it we see  $b^3(+\xi_0)$ . This is met by seven  $d^2$ -triangles  $d^2(+\xi_0, \text{space-time, space-time})$ . For expository purposes, we forget here about the fact that at  $\xi_0 = -1$  the triangles  $B^2(\beta, \dots)$  have void interiors. On the 2-sphere  $\partial b^3(+\xi_0)$  our triangles cut a connected graph which is displayed in the Figure 33.1. There, one should be able to read the continuous propagation of  $\vec{v}$  along  $P \times [-1 \leq \xi_0 \leq 0]$ .

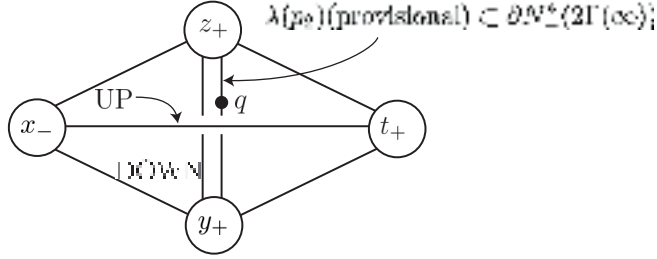
We start by looking now at the main, long green arcs  $\lambda(p_2)$  from 3), in Lemma 11.3. These arcs go from some  $u \in \{\text{original } c(b_i)\} \subset \partial N_+^4(2\Gamma(\infty))$  to some  $\{\text{little } \Gamma_j\} \subset \partial N_+^4(2\Gamma(\infty))$ , actually to  $p_2 \in \{\text{little } \Gamma_j\}$ . Figure 22 suggests the intersection  $\lambda(p_2) \cap \Delta^2 \cap [0 \geq \xi_0 \geq -1]$ . Figure 22 concerns a piece  $\lambda(p_2) \cap \text{body } d_k^2$  and, for the time being, we will stay in our proof with these pieces  $\lambda(p_2) \cap \text{body } d^2$ .

The idea now is to bring  $\lambda(p_2) \cap \text{body } d^2$ , modulo its end  $p_2 \in \partial N_+^4(\Gamma_1(\infty))$ , fully into  $\partial N_+^4(\Gamma_1(\infty))$ , by pushing it along the vector field  $\vec{v}$ . A priori two kinds of obstructions may occur at this point.

A) When inside a triangle  $d^2(u_1, u_2, u_3) \subset \partial N^4(P)$ , we may meet a curve  $[v_1, v_2] \subset \{\text{link}\}$  in the way. This obstruction which is a real one, is illustrated in the Figure 33. Look here at Figure 24-(A) for an explanation.

In Figure 33, the arc  $[x_-, t_+]$  (UP) obstructs the local push of  $\lambda(p_2)$  into  $\partial N_+^4(2\Gamma(\infty))$ . When we are completely outside of  $\Delta^2 \times [0 \geq \xi_0 \geq -1]$ , which certainly implies far from  $\Delta^2 \times (\xi_0 = -1)$ , then we can certainly apply the crossing freedom, change  $\text{UP} \Leftrightarrow \text{DOWN}$  around, and in this case, get rid of our obstruction.

We do not push the obstructing curve into  $\partial N_+^4(\Gamma_1(\infty))$ , as a priori we could, since we are not allowed to muck around with the confinement conditions (97). That would perturb the mechanism of section VII below. So we do **need** here an admissible crossing, coming with its change of local topology.



**Figure 33.**

A possible obstruction for pushing the long green arc  $\lambda(p_2)$  into  $\partial N_+^4(2\Gamma(\infty))$ . The  $q$  is here under  $p_2$  (Figure 22).

**But**, when we are at  $\xi_0 = 0$ , this admissible crossing would conflict with our policy concerning the LHS of the CLASP from Figure 22 (see here the 4.1) in Lemma 11.1. But then, in terms of Figure 30, when at  $P \times (\xi_0 = 0)$ , we have  $\{c(x_-, t_+)\} \subset \{\text{little } \Gamma_i \times (\xi_0 = 0)\} \subset \partial N_+^4(2\Gamma(\infty))$ , and so there is actually no obstruction to worry about. The **admissible** subdivision from Figure 30 is essential here as well as the fact that in Figure 30 the  $D^2(\Gamma_j \times (\xi_0 = -1))$  and  $D^2(\Gamma_j \times (\xi_0 = 0))$  get a similar treatment, although the  $D^2(\Gamma_j \times (\xi_0 = 0))$  carries NO ACCIDENT. Next, we look into the possible obstruction when we are at  $P \times (\xi_0 = -1)$  and  $+\xi_0$  lines could be in the way, in the same manner as the  $[x_-, t_+]$  is, in Figure 33. There is a unique CLASP coming with Figure 24; it goes along the  $\xi_0$ -line, being produced by the crossing (see Figure 22)

$$[x_-, t_+](\text{UP})/[z_+, y_+](\text{DOWN}).$$

With it comes an occurrence of  $\lambda(p_2)$ , at the level of Figure 24, along  $d^2(+\xi_0, y_+, z_+)$ . So, let us say that Figure 33 concerns now vertex  $P \times (\xi_0 = -1)$ , when  $p_2$  occurs in Figure 22. We let  $\lambda(p_2)$  go from  $b^3(+\xi_0)$  to  $b^3(y_+)$ , parallel and close to  $[\xi_0, y_+]$ , to begin with, and notice that the line  $[\beta, t_+]$  (Figure 24)  $\subset \partial N_+^4(2\Gamma(\infty))$  is NOT an OBSTRUCTION; see here the CLAIM (95).

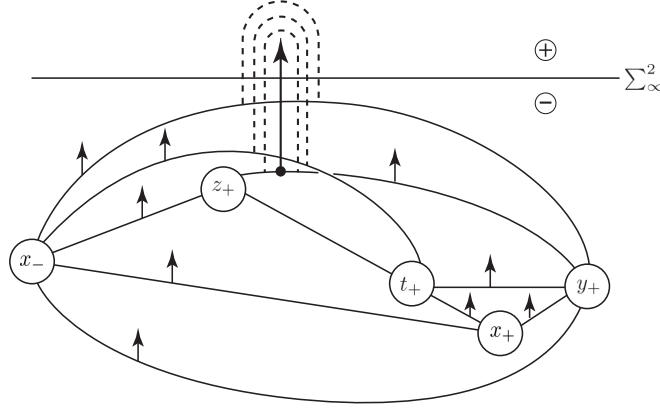
Then, in the triangle  $d^2(+\xi_0, y_+, z_+)$ , we let  $\lambda(p_2)$  go along the  $[y_+, z_+]$  (Figure 33 at  $P \times (\xi_0 = -1)$ ), now under

$$[x_-, t_+](\text{UP}) \subset \{\text{little } \Gamma_i \times (\xi_0 = -1) \subset \partial N_+^4(2\Gamma(\infty)),$$

NO OBSTRUCTION !, all the way up to  $p_2$ , at the level of the crossing, Figure 33.

All this takes care completely of the potential obstruction A).

B) But then, there is also a second OBSTRUCTION (still potential, so far), for pushing  $\lambda(p_2) \cap d_k^2$  into  $\partial N_+^4(2\Gamma(\infty))$ , namely one may find smooth sheets in body  $d^2$  or in body  $\mathcal{B}^2$ , in the way. But then, these can be pushed into  $\partial N_+^4(2\Gamma(\infty))$ , in front of the push of the  $\lambda(p_2)$ , without any harm. What we see going on in the Figure 33.1 is an instance of the general fact we have just stated.



**Figure 33.1.**

With  $P \times (\xi_0 = -1)$  like in the Figure 24, and with the  $b^3(+\xi_0)$  from that figure, we see here a generic section  $S^2(\xi_0)(0 \geq \xi_0 \geq -1)$  of

$$\partial N^4(\Gamma_1(\infty)) \mid [P \times [0 \geq \xi_0 \geq -1]] = \underbrace{\partial b^3(\xi_0)}_{S^2(\xi_0)} \times [0 \geq \xi_0 \geq -1].$$

#### Some explanations concerning Figure 33.1.

A) (Connection with Figure 24.) The Figure 24 is a projection on the plane of  $S^2_\infty$  with all the triangles being flat on that plane and the vector field  $\vec{v}$  sticking out of them. What we see here in Figure 33.1, is the way in which the triangles in question cut another 2-sphere, namely the  $\partial b^3(+\xi_0)$ . The  $S^2_\infty$  meets  $\partial b^3(+\xi_0)$  along the equator of  $\partial b^3(+\xi_0)$ .

B) (How to get this figure from Figure 24.) Imagine  $\partial b^3(+\xi_0)$  as another plane, perpendicular to the  $S^2_\infty$  from A). This *is* the plane of the present figure. Each triangle  $d^2(+\xi_0, u_1, u_2)$  from Figure 24 corresponds to an arc  $[u_1, u_2]$  here. What Figure 33.1 does, is to give the general idea of how one tackles the second obstruction B.

End of Explanations.

All this takes completely care of the pushing of  $\lambda(\text{green}) \cap d^2$  into  $\partial N^4_+(\Gamma_1(\infty))$  and we turn now to body  $\mathcal{B}^2$ . The body  $\mathcal{B}^2$  is made out of disjointed  $B^2$ -triangles, in the figures of type 9, with

$$B^2 = B^2(\beta, \pm \xi_0 \text{ or space-time}, \pm \xi_0 \text{ or space-time})$$

and these  $B^2$ -triangles are joined by rectangles going along the edges. Such a  $\beta$ -rectangle is displayed in Figure 22 and again in the Figure 34. The  $B^2$ -triangles can be seen in the Figures 24-(B) and 25-(B). Figure 34 displays a  $\beta$ -rectangle.

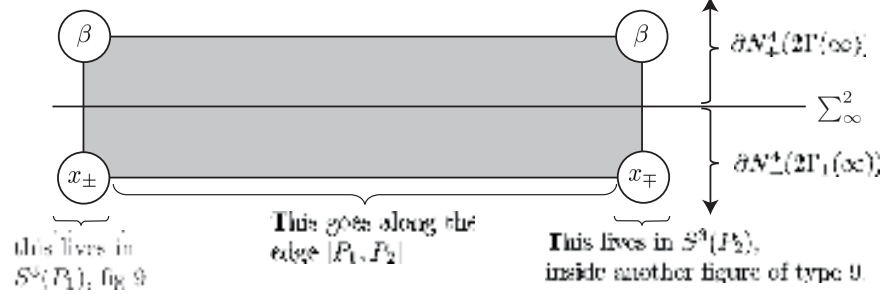


Figure 34.

We see here a  $\beta$ -rectangle going along the edge  $[P_1, P_2]$ . The body  $\mathcal{B}^2$  is made out of  $B^2$ -triangles and out of rectangles like this one. Each  $[\beta, u_\pm]$  occurs in some figure of type 9. In Figure 22 we have also displayed a  $\beta$ -rectangle, shaded, and going from  $+\xi_0$  to  $-\xi_0$ . The  $[\beta, x_\pm]$ 's are sides of  $B^2$ -triangles.

Of the three sides of a  $B^2$ -triangle, two are touching to the vertex  $\beta$  and, like  $\beta$ , they are in  $\partial N_+^4(2\Gamma(\infty))$ . They are arcs of the curves  $C(r \text{ or } b) \subset \partial N_+^4(2\Gamma(\infty))$ . The third side is contained in an arc [space-time, space-time] (or  $[\pm \xi_0, \text{space-time}] \subset \text{curve } C$  (generically  $\subset \partial N_-^4(2\Gamma(\infty))$ ). This is actually a  $\gamma_k^0 = \partial d_k^2$ . Figure 25-(B) illustrates the  $\Sigma_\infty^2 \cap \mathcal{B}^2$  and the transition from  $\mathcal{B}^2$  to the  $d_k^2$ 's, from  $(u, z_-, z_+, y_+)$  down. So, in Figure 25-(B) we have  $U \in \partial d_k^2$ .

In the figures of type 24, the  $B^2$ -triangles are higher (i.e. closer to the observer) than the rest. So there is no question of  $\lambda(\text{green}) \cap (B^2\text{-triangles})$  to be obstructed like in A).

But there is now another problem, namely we have transversal contacts like in Figure 28

$$y(p_2) \in \lambda(p_2) \cap \{\text{RIBBONS body } \mathcal{B}^2 \cap \text{body } d^2\}.$$

**Important Remark.** What we see in the Figure 28 suggests that the transversal contact  $y(p_2) \in \lambda(p_2) \cap \{\text{very special RIBBON}\}$ , is NOT an obstruction for getting to  $p_2$ . But what we are discussing now is not that, but the potential obstruction for pushing the  $\lambda(p_2)$  into  $\partial N_+^4(2\Gamma(\infty))$ . End of Remark.

With more details, the contact  $\lambda(p_2)(\text{provisional}) \mid \mathcal{B}^2 \cap \{\text{VERY SPECIAL RIBBON}\}$  (Figure 27), is displayed in Figure 25-(B). This concerns now a  $(B^2\text{-triangle}) \subset \text{body } \mathcal{B}^2$ , where the local piece of  $\lambda(p_2)$  which contains  $y(p_2)$  is localized, see here the Figure 25-(B).

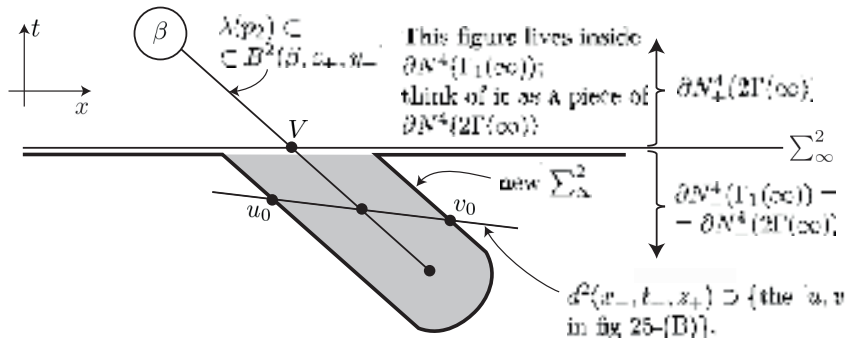


Figure 35.

The notations are here like in Figure 25-(B), with the proviso that, the present points  $u_0, v_0$ , are the  $\{u, v$  in Figure 25-(B) moved inside  $d^2$ , until  $[u_0, v_0] \subset \text{plane}(t, x)\}$ . The triangle  $B^2(\beta, z_+, y_+)$  from the 25-(B) lives in a plane which cuts transversally the present plane  $(t, x)$ , along the green line  $[\beta, U]$ . Here  $w = y(p_2)$ .

Figure 35, where for the time being one should ignore the fat line (which will correspond to a change  $\Sigma_\infty^2 \Rightarrow$  new  $\Sigma_\infty^2$ ), shows a plane  $(x, t) \subset \partial N^4(2\Gamma(\infty))$ , which cuts transversally both our triangles  $B^2(\beta, y_+, z_+) \subset$  body  $B^2$  and  $d^2(t_+, z_+, x_-) \subset$  body  $d^2$ , and also the SPLITTING SURFACE  $\Sigma_\infty^2$ . This plane  $(t, x)$  contains the line

$$(\lambda(p_2)(\text{provisional})) \mid [\beta, U], \text{ from Figure 25-(B).}$$

The point here, is that there is no obstruction, meaning NO sacro-sancted confinement conditions, in the way, for performing the following operation: Extend  $\partial N_+^4(2\Gamma(\infty))$  by a  $3^d$  dilatation **engulfing** the detail  $[(\beta), U] \cup [[u_0, v_0] \subset d^2]$  from Figure 35, with the dilatation in question concentrated around the shaded area in Figure 35.

Notice that we made use here of an engulfing process, rather than of some isotopy of curves.

[**Remark.** Would we have tried to push isotopically the

$$[\beta, U] \cup \underbrace{[u_0, v_0]}_w \text{ from Figure 35}$$

into  $\partial N_+^4(2\Gamma(\infty))$ , then we would have had to drag along appropriate neighbourhoods of it inside  $d^2(\beta, z_+, y_+) \cup d^2(x_-, t_+, z_+)$ , keeping at the same time track of boundary conditions. The engulfing is distinctly more economical.]

With this engulfing, which redefines our basic SPLITTING and which hence changes  $\Sigma_\infty^2$  into  $\Sigma_\infty^2$  (new), we have now

$$[(\beta), U] \subset \partial N_+^4(2\Gamma(\infty)).$$

All this takes completely care of the  $\sum_{p_2 \in \{\text{punctures body } \delta^2 \cap \Gamma_i\}} \lambda(p_2) \subset \partial N_+^4(2\Gamma(\infty))$ , which has gotten into  $\partial N_+^4(2\Gamma(\infty))$ , where we wanted it to be.

So now we have to take care of the  $\lambda(q)$ 's too.

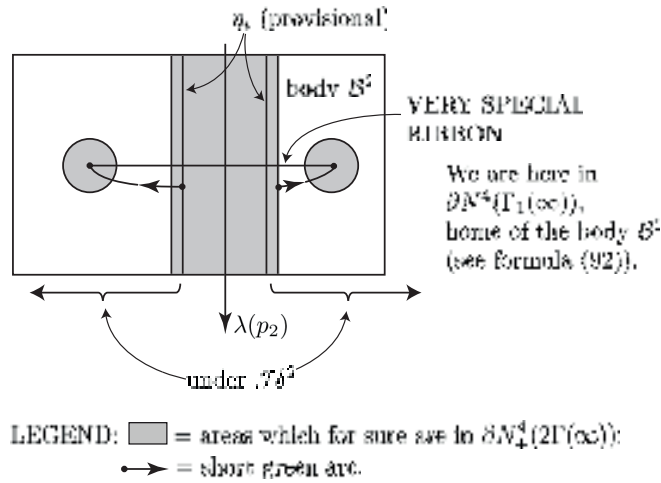


Figure 35.1.

Compare this with the Figure 28-(II). The  $\lambda(p_2)$  has already gone through, by now.

By combining Figures 28 with 35.1, one sees that there is no obstructions for pushing the short green arcs  $\lambda(q_1), \lambda(q_2)$  into  $\partial N_+^4(2\Gamma(\infty))$ . What one sees in Figure 35.1 (and look at 28-(II) too) is what happens along a VERY SPECIAL RIBBON after  $\lambda(p_2)$  has gone through  $y(p)$ . We can go then step by step to the whole component of special RIBBONS (Figure 28) and push all the corresponding short green arcs  $\lambda(q) \subset \text{body } \mathcal{B}^2$  into  $\partial N_+^4(2\Gamma(\infty))$ ; there is no obstruction in the way.

By now we have gotten at level of (93.1), with the  $\Sigma c(b_i)$  pushed over the  $p_2$ 's and with the corresponding accidents all destroyed. Let us say that what we have realized is something which goes beyond the (93.1) and where the (98) is now with us too. But we keep the same notations as in (93.1), and our final result is

$$(98.1) \quad \left( \sum_1^M \delta_i^2, \sum_1^M \partial \delta_i^2 = c(b_i)(\text{initial}) \right) \implies (93.1) \implies \left( \sum_1^M \delta_i^2, \sum_1^M c(b_i)(\lambda) = \partial \delta_i^2 \right), \text{ when now we also have } \sum_1^M c(b_i)(\lambda) \subset \partial N_+^4(2\Gamma(\infty)).$$

Lemma 12 is now proved.

**Theorem 13. (The little blue diagonalization)**

1) For any long green arc  $\lambda(p_2)$  we have a decomposition into two successive pieces  $\lambda(p_2) = (\lambda(p_2) \cap \mathcal{B}) \cup (\lambda(p_2) \cap d_k^2)$ , with  $(\lambda(p_2) \cap \mathcal{B})$  starting at  $c(b_i)$  and with  $\lambda(p_2) \cap d_k^2$  ending at  $p_2$ . With this, at the price of complicating the geometric intersections matrices  $\Gamma_j \cdot B$  and  $\Gamma_j \cdot R$  we can achieve that

$$(99) \quad (\lambda(p_2) \cap d_k^2) \cdot B = \emptyset.$$

This operation does not touch  $C \cdot h$  and hence it does not modify the topology of LAVA.

1-bis) The step leading to (99) can also be read as an operation which leaves the proper embeddings  $B, R \subset N^4(2\Gamma(\infty))$  unchanged, but which modifies the  $\sum_1^{\bar{n}} \Gamma_j \subset \partial N^4(2\Gamma(\infty))$ , inside its isotopy class.

2) We also have

$$(100) \quad (\lambda(p_2) \cap \mathcal{B}) \cdot B \subset \{\text{very trivial } B\text{'s, in the sense of (94.2)}\}.$$

3) When it comes to the short green arcs  $\lambda(q)$  we also have, like in 2) above,

$$(100.1) \quad \lambda(q) \cdot B \subset \{\text{very trivial } B\text{'s, in the sense of (94.2)}\}.$$

4) (Reminder) We know already that the system of discs (98.1), which I will re-write here for the convenience of the reader

$$(100.2) \quad \left( \sum_1^M \delta_i^2, \sum_1^M \partial \delta_i^2 = c(b_i)(\lambda) \right) \xrightarrow{J} (\partial N^4(2X_0^2)^\wedge \times [0, 1], \partial N^4(2X_0^2)^\wedge \times \{0\}),$$

where  $\partial N^4(2X_0^2)^\wedge \times [0, 1] \subset N_1^4(2X_0^2)^\wedge = N^4(2X_0^2)^\wedge \cup (\partial N^4(2X_0^2)^\wedge \times [0, 1])$  has the following features.

a) It is a smooth embedding without any ACCIDENT, i.e. it is a system of discs which is both embedded and external to

$$N^4(2X_0^2)^\wedge \subset N_1^4(2X_0^2).$$

b) We have here (and see (98.1))

$$\sum_1^M c(b_i)(\lambda) \subset (\partial N_+^4(2\Gamma(\infty))) \cap \partial N^4(2X_0^2).$$

5) *There is a transformation*

$$(101) \quad \sum_1^M c(b_i)(\lambda) \implies \sum_1^M \eta_i(\text{green}) \subset \partial N_+^4(2\Gamma(\infty)) \cap \partial N^4(2X_0^2)$$

which is *CONFINED* inside  $\partial N_+^4(2\Gamma(\infty))$ , and which drags the cobounding  $\sum_1^M \delta_i^2$  along, so that at the end of (101), with the redefined  $\sum_1^M \delta_i^2$ , we have, like in (100.2)

$$\left( \sum_1^M \delta_i^2, \sum_1^M \eta_i(\text{green}) = \partial \delta_i^2 \right) \xrightarrow{\text{EMBEDDING}} (\partial N^4(2X_0^2)^\wedge \times [0, 1], \partial N^4(2X_0^2) \times \{0\}).$$

Moreover, now the condition stated in (82.1) is finally satisfied, i.e. we have

$$(101.1) \quad \eta_i(\text{green}) \cdot b_j = \delta_{ij} \text{ for } 1 \leq i, j \leq M \text{ AND, at the same time}$$

$$\eta_j(\text{green}) \cdot \left( B_1 - \sum_1^M b_i \right) = 0.$$

6) *At this point, we can finally write down the FORCED CONFINEMENT CONDITIONS, superseding from now on the (97), with all the curves, internal and external, presented in their full glory*

$$(101.1\text{-bis}) \quad \sum_i \{C_i \text{ remaining}\} + \sum_j \{\Gamma_j \text{ remaining}\} + \sum_k \gamma_k^0 \subset \partial N_-^4(2\Gamma(\infty)),$$

AND

$$\sum_i \{\text{little } C_i\} + \sum_j \{\text{little } \Gamma_j\} + \sum \eta_i \times b + \sum c(r) \subset \partial N_+^4(2\Gamma(\infty)) \supset \sum_1^M \eta_i(\text{green}).$$

**Proof.** We start by reviewing the GPS structure of  $X^4$  (see (6)), which extends in an obvious way to  $X^2[\text{new}]$ . And right now we will look into the details of the BLUE collapsing flow.

On the piece  $(\Gamma(1) \times [0 \geq \xi_0 \geq -1]) \cup (\Delta^2 \times (\xi_0 = -1))$ , things are already settled, since all the  $D^2(\Gamma_i)$ 's are  $D^2(\gamma^1)$ 's, BLUE-wise, and the BLUE 2<sup>d</sup> collapse proceeds by  $\Gamma(1) \times [-1 \leq \xi_0 \leq 0] \searrow \Gamma(1) \times (\xi_0 = 0)$ , a.s.o.

Next, we will take another close look at the BLUE flow on  $X^2(\text{old}) \subset X^2[\text{NEW}]$ . The general strategy for this flow is presented in Figure 35.2, and details will follow.

Here is the description of the 2<sup>d</sup> BLUE collapsing flow inside  $X^2$  (GPS), as illustrated in Figure 35.2. We insist only on the UPPER-LEFT corner of the figure.

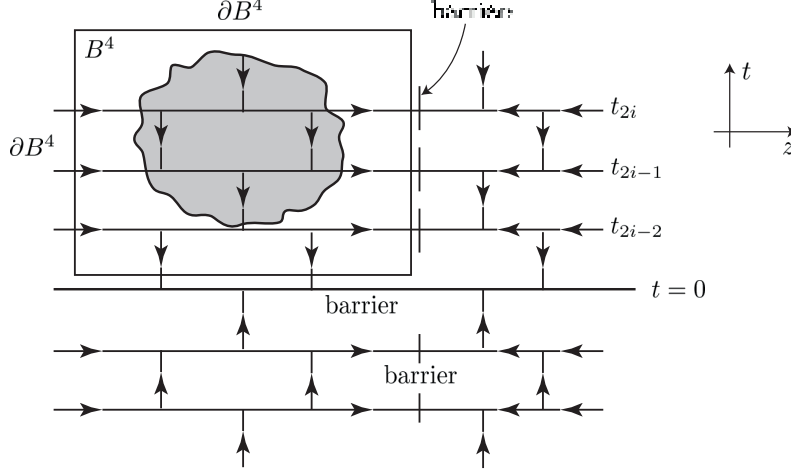
(101.2) A) On each  $X^3 \times t_j$  the trajectories are linear, in the  $z$ -direction. Any horizontal 2<sup>d</sup> plaquettes ( $z = \text{const}$ ) is a  $D^2(\gamma^1)$  which gets deleted (before any 2<sup>d</sup> collapse can start).

B) Inside  $X^3 \times t_i$ , the linear 2<sup>d</sup> BLUE collapsing flow, which goes in the  $z$ -direction, may meet horizontal ( $z = \text{const}$ ) edges  $e \subset 2X^1(\text{BLACK}) \cup 2X^1(\text{ORANGE})$ . At this point, the combination

$$(*_1) \quad (\text{colour of } e) \times (\text{parity of } i),$$

decides if  $e \times [t_i, t_{i\pm 1}] \subset X^2$  (see (6)), and these 2-cells  $e \times [t_i, t_{i\pm 1}]$  are declared to be  $D^2(\gamma^1)$ 's. Their interiors are to be **deleted**, like for the  $D^2(\gamma^1)$ 's in A) above, before any 2<sup>d</sup> BLUE collapse can start.





**Figure 35.2.**

The general strategy for the 2<sup>d</sup> BLUE collapsing flow on  $X^2(\text{old})$  (GPS). This is, of course, a very schematical figure.

LEGEND:  $\rightarrow\leftarrow = 2^d$  flow inside  $X^3 \times t_j$ ;  $\uparrow\downarrow\uparrow\downarrow = 2^d$  flow inside  $[t_j, t_{j+1}]$ ; —, | = BARRIERS. The flow can be fixed on the barriers too, but we will do that only if and when needed. Importantly, it is only the upper-left corner, with  $\Delta_{\text{Schoenflies}}^4 \subset B^4$  which really concerns us. The  $\partial B^4$  rests on the BARRIERS. Vertical arrows starting at a  $t_{\text{odd}} > 0$  or  $t_{\text{even}} < 0$  are ORANGE, all the others are BLACK.

**[Remark.** We are a bit cavalier concerning these  $D^2(\gamma^1)$ 's. Normally, there should be a 3<sup>d</sup> BLUE collapsing flow, prior to the 2<sup>d</sup> one, which should demolish them. But we are in a purely 2<sup>d</sup> context, so we demolish the  $D^2(\gamma^1)$ 's by decree. If it would be there, the 3<sup>d</sup> collapsing flow would also need its 3<sup>d</sup> BARRIERS. We can assume them far from our interesting UPPER-LEFT corner.]

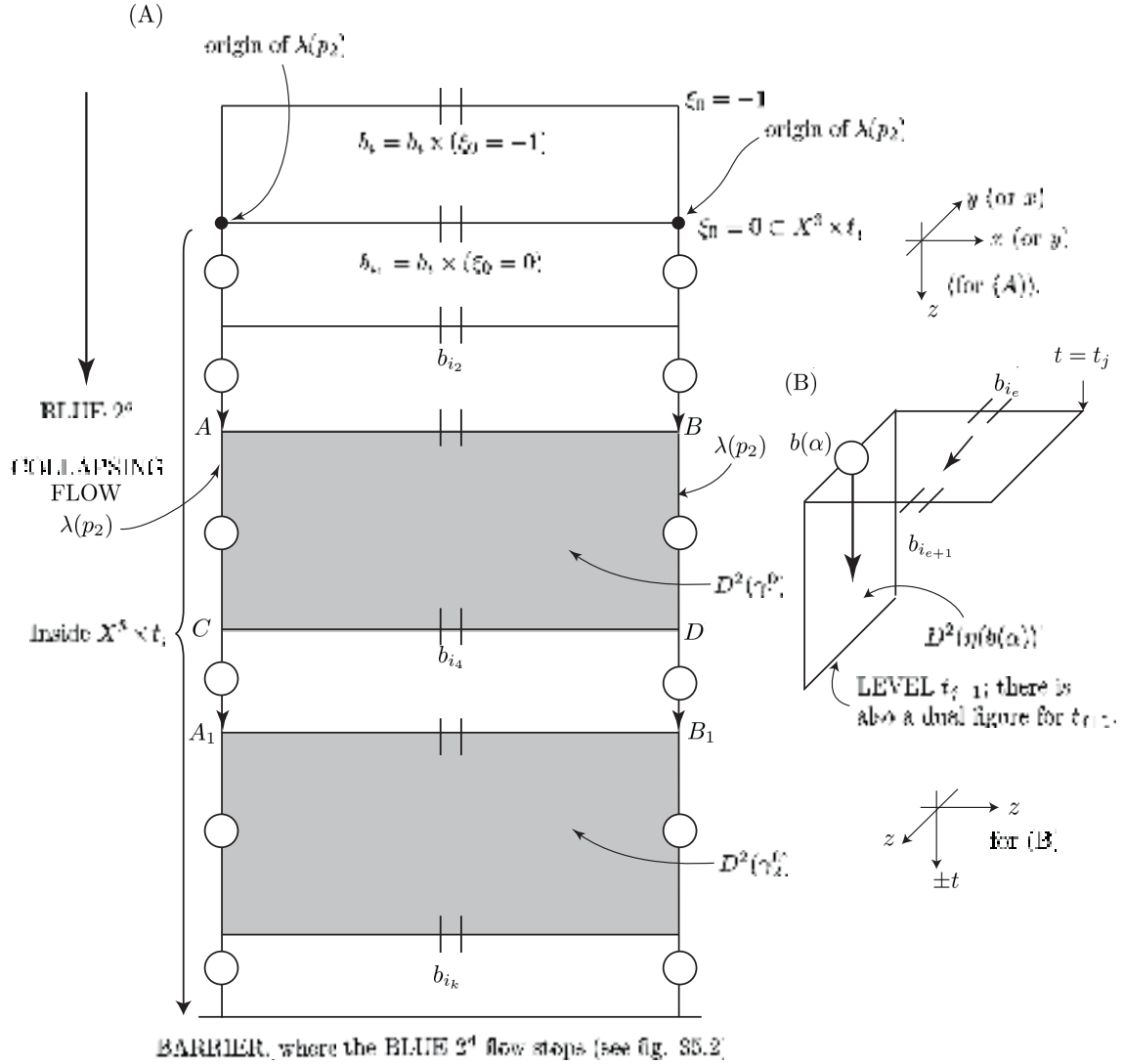
C) After all the int  $D^2(\gamma^1)$ 's are deleted, the BLUE 2<sup>d</sup> collapse takes place, and it leaves us with a residual graph  $\Gamma_{\text{residual}} \subset (X^1(\text{BLACK}) \cup X^1(\text{ORANGE})) \mid t_i$ . The common part of the two  $X^2$ 's is generated by things like  $a, f, d$ , in the Figure 5-(B). With these things, we consider now

$$(*)_2 \quad \Gamma_{\text{residual}} \cap (2X^1(\text{BLACK}) \cup 2X^1(\text{ORANGE})) \mid t_i.$$

Any edge  $e \in (*)_2$  goes in the  $z$ -direction and carries a  $b_0 \in B_0$ ; the BLUE 2<sup>d</sup> collapsing flow from  $b_0$  goes in the direction  $\pm t$ , along  $[t_i, t_{i\pm 1}]$ , by the  $(*)_1$ -dependent rules of Figure 35.2. All these  $b_0$ 's are very trivial, in the sense of (94.2).

Concerning our same 2<sup>d</sup> BLUE collapsing flow, inside  $\Gamma(1) \times [0 \geq \xi_0 \geq -1]$  it follows the direction  $+\xi_0$  and all the  $D^2(\Gamma_i) \times (\xi_0 = -1)$ 's are  $D^2(\gamma^1)$ 's. All this clinches the definition of the 2<sup>d</sup> BLUE flow for  $X^2[\text{new}]$  (GPS). On the other hand, the RED 2<sup>d</sup> collapsing flow depends on how the  $\Delta_{\text{Schoenflies}}^4$  is located inside  $X^4$ , and so it cannot be presented in a similar cavalier and explicit manner. But then, see here the PL-Lemma 3.1 too.

Once the structure of the 2<sup>d</sup> BLUE collapsing flow has been completely unrolled, with its linear **trajectories**, we will look now at the BLUE 2<sup>d</sup> collapsing trajectory of generic  $b_i \in \Gamma(1) \times (\xi_0 = -1)$ . This is displayed in Figure 35.3, where one also sees the  $\lambda(p_2) \cap \mathcal{B}$ .



**Figure 35.3.**

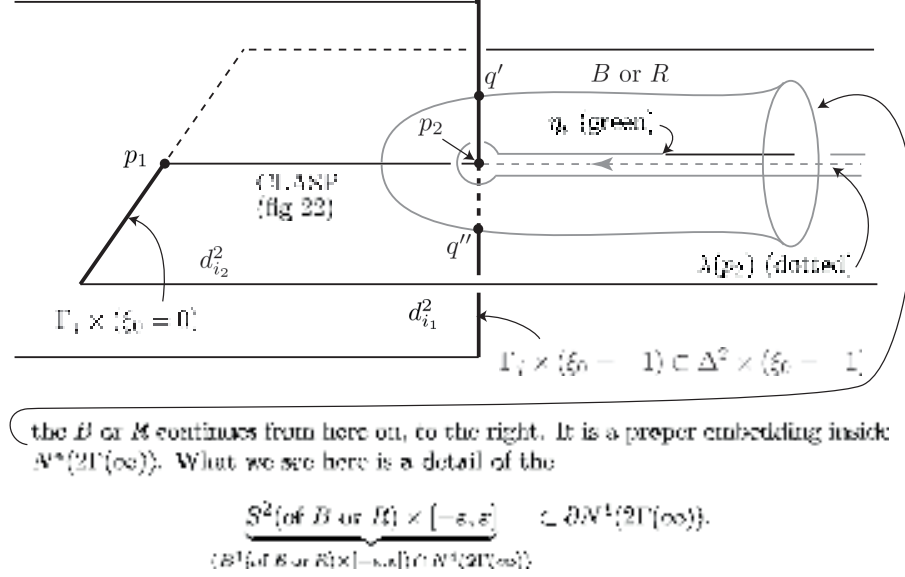
LEGEND:  $\blacksquare = D^2(\gamma_k^0)$ ;  $\bigcirc = b(\alpha) \in B_0(?)$ . This  $b(\alpha)$  may be or not be in  $B_0$ , depending on the combination  $(*_1)$ ;  $\longrightarrow = \lambda(p_2) \cap \mathcal{B}^2$ . It has several parallel strands, and at  $A, B, A_1, B_1$ , it leaves  $\mathcal{B}^2$ .

Forgetting momentarily about the  $b(\alpha)$ 's, (which are very trivial element of  $B$ ), our trajectory of  $b_i = b_i \times (\xi_0 = -1)$  is  $b_i \rightarrow b_{i_1} \rightarrow b_{i_2} \rightarrow \dots$ , realized by a vertical, linearly ordered column of  $2^d$  plaquettes  $D^2(\eta)$ . Some of them come with  $[D^2(\eta)] = D^2(\gamma^0)$ , shaded in Figure 35.3-(A), and then the piece  $\Sigma d_k^2 \subset \delta_i^2$  starts.

The  $b(\alpha)$ 's are the potential sites corresponding to a  $b_0 \in B_0$ , living on an edge  $e \in (*_2)$ , produced by the appropriate combination  $(*_1)$ , see here (101.2). More explicitly, if the corresponding edge  $e$  is in

$(2X^1(\text{BLACK}) \cup 2X^1(\text{ORANGE})) \cap \Gamma_{\text{residual}}$  then, depending on the combination COLOUR/parity of  $j$  (occurring in  $t_j$ ), we have actually a  $b \in B_0$  at  $b(\alpha)$ . I repeat that these  $b(\alpha)$ 's are very trivial  $B$ 's (see (94.2)).

On  $c(b_i) \cap (\xi_0 = 0)$  we see two green fat points ("origin of  $\lambda(p_2)$ ") from which green arcs start to each  $D^2(\gamma^0) = [D^2(\eta)]$ . These arcs may have pieces in common, that is OK. These green arcs, seable in Figure 35.3, are  $\lambda(p_2) \cap B$ 's.



**Figure 35.4.**

This figure which lives inside  $\partial N^4(\Gamma_1(\infty))$  is to be compared to the Figures 20, 22, 23. The contact

$$(\lambda(p_2) \cap d_k^2) \pitchfork \{\text{cocore } B \text{ or } R\}$$

is destroyed by pushing the cocore in question over the puncture  $p_2 \in \Gamma_j \times (\xi_0 = -1)$ . This will create new intersection points

$$q', q'' \in (\Gamma_j \times (\xi_0 = -1)) \cap \{\text{cocore } B \text{ or } R\},$$

which increase the corresponding geometric intersection matrices. Here only a piece of the  $\{\text{cocore } B \text{ or } R\}$  is visible. It is a 2-cell and there are many parallel copies of it, in order to get the full picture around  $p_2$ . But then each such 2-cell continues to a whole copy of

$$S^2 \subset S^2 \times [-\varepsilon, \varepsilon] \subset \partial N^4(2\Gamma(\infty)),$$

continuing with a 1-handle  $(B^3 \times [-\varepsilon, \varepsilon]) \subset N^4(2\Gamma(\infty))$  (BLUE or RED).

We move now to the point 1) in our Theorem 13. A priori we have transversal contacts

$$(102) \quad (\lambda(p_2) \cap d_k^2) \pitchfork B$$

and these (102) are intermingled along the  $\lambda(p_2) \cap d_k^2$  with similar contacts

$$(102.1) \quad (\lambda(p_2) \cap d_k^2) \pitchfork R.$$

The IDEA now is to PUSH THE (102) + (102.1) OVER THE PUNCTURE  $p_2 \in \Gamma_j \times (\xi_0 = -1)$ . Here are the various noteworthy items concerning this STEP, which is displayed graphically in the Figure 35.4.

A) All the punctures  $p_2$  occur on curves  $\Gamma_j \times (\xi_0 = -1) \subset \Delta^2 \times (\xi_0 = -1)$ , making that our step does not change the matrix  $C \cdot h$  and hence it does not touch the LAVA which has its topology stay intact. It does not touch the geometric intersection matrix  $\eta \cdot \beta$  either, since both at the levels  $X^2[\text{NEW}]$  and  $2X_0^2$ ,  $D^2(\Gamma_j \times (\xi_0 = -1))$  is always a  $D^2(\gamma^1)$ , never a  $D^2(\eta)$ .

B) Like in Figure 35.4, we create contacts  $q', q'' \in (\Gamma_j \times (\xi_0 = -1)) \cap \{\text{cocore of } B \text{ or } R\}$ , actually  $q', q'' \in \{\text{little } \Gamma_j \times (\xi_0 = -1)\} \cap \{\text{cocore } B \text{ or } R\} \subset \partial N_+^4(2\Gamma(\infty))$ .

These do NOT OBSTRUCT anything; I mean they leave  $\Gamma_j \times (\xi_0 = -1) \subset \partial N^4(\Gamma_1(\infty))$  located just as before. The only thing they change are the items  $(\Gamma_j \times (\xi_0 = -1)) \cdot B$  and  $(\Gamma_j \times (\xi_0 = -1)) \cdot R$  in the geometric intersection matrices, and this is harmless.

C) The way we presented things in Figure 35.4 is to complicate the picture of the  $\{\text{cocore}(B, R)\}$ , leaving the curves  $\Gamma_j$  in peace. But there is also another way of describing the same step. Leave the triplet

$$(N^4(2\Gamma(\infty)); \text{cocores of } B, \text{cocores of } R)$$

completely unchanged; i.e. do not modify Figure 46 which will be crucial for our future COLOUR-CHANGING. BUT then we start by **shortening** drastically the  $\lambda(p_2) \cap d_k^2$ , starting at  $p_2$ , push the  $\Gamma_j \times (\xi_0 = -1)$  through our  $\{\text{cocores } R \text{ and } B\}$ , until the  $p_2$  curves now very close to the point in  $\partial d_k^2 \subset \text{int } \delta_i^2$  where we find  $\lambda(p_2) \cap \partial d_k^2$ . This clinches the proof of our 1).

The point 2) is readable in the Figure 35.3 when  $\lambda(p_2) \cap \mathcal{B}$  is concentrated on the two lateral sides, avoiding the  $b_{i_j}$ 's and only meeting the very trivial  $b(\alpha)$ 's. The linearity of the BLUE collapsing trajectories plays here.

We move now to 3) which concerns the short  $\lambda(q) \subset \text{body } \mathcal{B}^2 \subset \delta_i^2$ . There is here the obvious potential danger, readable in the Figure 25-(B): We may need to propagate  $\lambda(q)$  from  $p$  to  $q$ , over the edge  $[P_1, P_2]$  and **if** this edge is occurring now explicitly in Figure 35.3 and is horizontal, like  $[AB]$ , the  $[P_1 P_2]$  contains one of the non trivial  $b_{i_k}$ 's from the BLUE collapsing trajectory of  $b_i = b_i \times (\xi_0 = -1)$ . At this point, we have to look carefully at the precise manner in which the  $\delta_i^2$  is being put up at point 1) in Lemma 10, from the successive pieces  $B_{i_k}^2$  (and  $B^2(b(\alpha))$ , corresponding to Figure 35.3. Here, when the  $B_{i_k}^2$  and  $B_{i_{k+1}}^2$  are put together, then the common collar  $c(b_{i_{k+1}}) \times [0, \varepsilon]$  gets deleted.

So, let us consider the potentially dangerous situation when  $\lambda(q)$  goes along the  $[p, q]$  in Figure 25-(B), with (and see here the notations from Figure 25-(B))  $b_{i_{k+1}} \in [P_1, P_2]$ . I CLAIM that, actually, this  $\lambda(q)$  is **not physically present in  $\delta_i^2$** , and hence we do not have to worry about it.

**[Proof of the Claim.** We work now with the decomposition  $\delta_i^2 = \mathcal{B}_i^2 \cup \Sigma d_k^2$ 's and the  $\mathcal{B}_i^2$  is completely readable from Figure 35.3, which contains all the necessary ingredients. We have

$\lambda(q) \mid [P_1, P_2] \subset \{\text{the two triangles from Figure 25-(B), which live, respectively in } \partial N^4(P_1), \partial N^4(P_2)\} \cup \{\text{The } b\text{-rectangle } [P_1, P_2]\} \subset \{\text{the collar zone } c(b_{i_{k+1}}) \times [0, \varepsilon]\}.$

And the collar zone in question gets deleted when we put up  $\delta_i^2$ . This proves our CLAIM.]

All this shows that the  $\lambda(q)$ 's finding themselves naturally in the rectangles

$$\{\text{horizontal edge like } [A, B], \text{ in Figure 35.3}\} \times [r, b]$$

are not actually in  $\mathcal{B}_i \subset \delta_i^2$ .

So, the façade of  $\mathcal{B}_i^2$ , which is presented explicitly in Figure 35.3, comes with no problem for  $\lambda(q) \cdot B$  in 3). We move then to the

$$(**) \quad \{\text{two lateral vertical sides of Figure 35.3}\} \times [r, b],$$

which of course is not visible in the figure in question. In that region the only contacts  $\lambda(q) \cdot B$  which we may find are with the  $b(\alpha)$ 's which is OK. [Notice, incidentally, that the procedure from Figure 35.4, via which we have demolished the contacts

$$(\lambda(p_2) \cap d_k^2) \cap (B \text{ and } R), \text{ which concerned the } \Gamma_j \times (\xi_0 = -1),$$

could never work for  $\lambda(q) \cap R$  for which the COLOUR-CHANGING mechanism is certainly necessary; remember that the  $\lambda(q)$ 's also contribute to  $\eta_i(\text{green})$ .]

Point 3) in our Lemma has by now been proved. Also our  $\lambda$ 's are all there and point 4) is with us too. So we go to point 5).

Each  $c(b(\alpha))$  bounds the disc  $B^2(b(\alpha)) \subset 2X_0^2$ , from (79.0). Moreover we have as the only contacts of  $c(b(\alpha))$  with  $B_1$  the following ones

$$(102.2) \quad c(b(\alpha)) \cdot b(\alpha) = 1 = c(b(\alpha)) \cdot (b(\alpha) \times b).$$

Along the  $b(\alpha)$ , the two loops  $c(b(\alpha))$  and  $c(b_i)(\lambda)$  touch, when  $\lambda(p_2) \cap b(\alpha) \neq \emptyset$  and  $\lambda(p_2) \subset \delta_i^2$ . From here on, we proceed via the following steps.

i) Making use of (102.2), the contact  $c(b_i)(\lambda) \cdot b(\alpha)$  can be changed into  $c(b_i)(\lambda) \cdot (b(\alpha) \times b)$ , dragging of course the  $\delta_i^2$  along, through an ambient isotopy of  $\partial N^4(2X_0^2)^\wedge \times [0, 1]$ . What we have gained via this step is that now the only off-diagonal contacts  $c(b_i)(\lambda) \cdot B_1$  are exactly the following

$$(102.3) \quad c(b_i)(\lambda) \cdot (b_i \times b) = 1, \quad c(b_i)(\lambda) \cdot (b(\alpha) \times b) = 1,$$

both living on the  $X_b^2$ -side.

ii) We use now our BLUE geometric intersection matrix, which has been transported in the  $X_b^2$ -side,

$$\eta \cdot B \mid 2X^2 = \text{easy id} + \text{nilpotent}$$

and get rid of the off-diagonal terms (102.3). This last step is completely concentrated on the  $X_b^2$ -side, like i) too. All this also clinches the transformation

$$c(b_i) \implies c(b_i)(\lambda) \implies \eta_i(\text{green})$$

which is all confined inside  $\partial N_+^4(2\Gamma(\infty))$ . Lemma 12 is, of course, used here. End of the Proof of Theorem 13.

**Remark.** Notice that it is on the (\*\*) above (i.e. the  $\{\text{lateral vertical sides of Figure 35.3}\} \times [r, b]$ ), that the mechanism from the Figures 28, 29, occurring in the proof of Lemma 11.3 come fully into play.

## 6 The balancing of red and blue, and the abstract theory of colour-changing

We will consider an infinite connected graph  $\Gamma$ . In real-life this will be  $\Gamma(\infty) = \Gamma(\infty) \times r$  or  $\Gamma(2\infty)$ , but let us be, for a while, a bit more general. By definition, a discrete set  $E \subset \Gamma$  has the ***P-property*** if  $\Gamma - E$  is a tree. An edge  $e \in \Gamma$  contains at most one  $x \in E$  and we will often identify  $x$  and  $e \ni x$ , thinking also of  $x$  as being a 1-handle (attached to  $\Gamma - E \approx \text{pt}$ ).

We are given two discrete subsets  $R \subset \Gamma \supset B$  each endowed with the *P-property*. It is also understood that, when  $x \in e \ni y$ , then  $x = y$ , leaving us hence with the partitions

$$R = (R - B) + (R \cap B), \quad B = (B - R) + (R \cap B).$$

**Lemma 14. (The abstract colour-changing process.)** *We can get a bijection*

$$(103) \quad R \xrightarrow[\approx]{\Phi} B, \text{ which is id on } R \cap B,$$

*via the following INDUCTIVE PROCESS.*

**Lemma 14. (The abstract colour-changing process.)** *There is a bijection*

$$(103) \quad R \xrightarrow[\approx]{\Phi} B, \text{ which is id on } R \cap B,$$

*gotten via the following INDUCTIVE PROCESS.*

1) Since  $R$  has property  $P$  for  $\Gamma$  and  $(\Gamma - R \cap B) - (R - B) = \Gamma - R$ , it follows that  $R - R \cap B$  has Property  $P$  for  $\Gamma - R \cap B$  and then, similarly  $B - B \cap R$  also has property  $P$  for the same  $\Gamma - R \cap B$ . With this,  $B \cap R$  will be just **mute** in what follows next and we will just work with  $\Gamma \equiv \Gamma - R \cap B$ ,  $R - B \cap R$  and  $B - B \cap R$ . Next, let us pick some arbitrarily chosen  $b(1) \in B - R$ ; since  $R \in P$ , we have

$$(104) \quad \Gamma - R = X_1 \cup b(1) \cup Y_1,$$

where  $X_1, Y_1$  are two disjointed trees, joined together by  $b(1)$ .

2) With all these things, there has to be an  $r(1) \in B - R$  which joins  $X_1$  and  $Y_1$  too. We define then  $\Phi(r(1)) = b(1)$  and I claim that the property  $P$  is true for  $R(1) \equiv R - r(1) + b(1)$ . [In this little story, the fact that  $\Gamma - R$  is a tree forces the existence of  $b(1)$  for (104), while the fact that  $\Gamma - B$  is tree, forces the existence of  $r(1)$ . There is a lot of arbitrariness in the way the map  $\Phi$  is constructed, but this will turn out to be OK.]

3) Assume, inductively, that continuing this, for  $n = 1, 2, \dots, j$  we have already managed to define a map  $r(n) \xrightarrow{\Phi} b(n)$  (bijection for  $n \leq j$ ), and that the  $R(j) \equiv R - \{r(1), r(2), \dots, r(j)\} + \{b(1), b(2), \dots, b(j)\}$  has Property  $P$ .

Notice here that  $R(j) = R(j-1) - r(j) + b(j)$ .

We will pick up some  $b(j+1) \in B - R(j)$  and with this, we get a decomposition which is like in (104), but now at a higher level, namely the

$$(105) \quad \Gamma - R(j) = X_{j+1} \cup b(j+1) \cup Y_{j+1}.$$

There has to be some  $r(j+1) \in R(j) - B$  connecting  $X_{j+1}$  to  $Y_{j+1}$ . Then we set, by definition  $\Phi(r(j+1)) = b(j+1)$  and this kind of process continues now indefinitely. This **is** our INDUCTIVE PROCESS.

4) And this process gives us a map

$$R \supset B \cap R + \sum_1^{\infty} r(n) \xrightarrow{\Phi} B$$

which is surjective on  $B$  and also injective on its domain of definition.

5) But I **claim** that we also have  $R \cap B + \sum_1^{\infty} r(n) = R$ , and this clinches then our (103). The  $\Phi$  above is a bijection.

With this our statement of Lemma 14 is ended, but let us notice, right away, that there is a slightly different way of formulating 4) + 5). So, let us restart from the end of 3) when the INDUCTIVE PROCESS has already been unrolled.

From this on, our lemma says the following things (and whenever appropriate, the  $R, B$ 's may mean here  $R - B, B - R$ ).

6) This process can be continued until all of  $B$  is exhausted. When that point has been reached we have gotten, for  $j = \infty$ , a set

$$R(\infty) = R - \{\text{a certain subset } R_0 \subset R\} + B.$$

7) We have  $R(\infty) = B$ . Hence, also,  $R_0 = R$  and at the grand end of the INDUCTIVE PROCESS, all the red elements of  $R$  **have turned** BLUE.

This ENDS the alternative way of stating Lemma 14.  $\square$

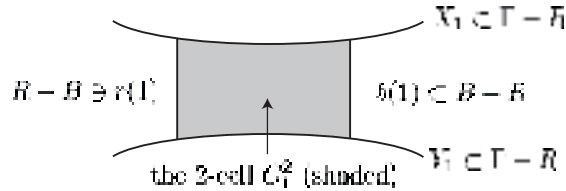
**Remarks.** A) Of course, so far all this is a purely ABSTRACT story, without any geometrical counterpart. But that geometric counterpart will come soon. And, in four dimensions, via handle-sliding and LAVA MOVES, we will be able to play a sufficiently high truncation of the  $R(\infty) = B$  from 7) above, so that we should get the BIG BLUE DIAGONALIZATION

$$\eta_j(\text{green}) \cdot \{\text{extended cocore } (b_i)\}^\wedge = \delta_{ij}, \quad \text{for } 1 \leq i, j \leq M.$$

B) In the process, the  $C \cdot h = \text{id} + \text{nilpotent}$  will get VIOLATED, but the LAVA MOVES will be such that the PRODUCT PROPERTY OF LAVA will always stay with us, and it is via that property that the extended cocores will be defined. End of Remarks.

**Proof of Lemma 14.** In the Figure 36 we have represented, schematically, the decomposition (104). In the same figure, we introduce the single closed loop  $\partial C_1^2 \rightarrow \Gamma$ , boundary of the purely abstract 2-cell  $C_1^2$ , which is such that

$$\partial C_1^2 \cdot r(1) = 1 = \partial C_1^2 \cdot b(1), \quad \text{and} \quad \partial C_1^2 \cdot (R - r(1)) = 0 = C_1^2 \cdot (R(1) - b(1)).$$



**Figure 36.**

Illustration for (104).  $\Gamma$  really means here  $\Gamma - R \cap B$ . The  $\partial C_1^2$  is a simple loop and  $C_1^2$  (shaded) is abstract. The position of the subsets  $\{r(1) + b(1)\} \cap X_1$  and  $\{r(1) + b(1)\} \cap Y_1$  determines how, in our drawing, we orient  $X_1, Y_1$  with respect to each other.

We will prove now that  $R(1) \in P$  and the same kind of arguments will work for  $R(j) \in P$  too. The set  $R$  is a free basis for  $H_1(\Gamma, \text{rel}(\Gamma - R)) = H_1 \Gamma$ . We introduce the abstract 2-cell  $C_1^2$  like in Figure 36. Then, in

$$\begin{array}{ccc} \Gamma - b(1) \subset \Gamma & \xrightarrow{p_1} & \Gamma \cup C_1^2 \equiv \Gamma \cup_{\partial C_1^2} C_1^2 \\ \downarrow f_1 & & \uparrow \end{array}$$

the homology class  $[r(1)] \in H_1 \Gamma$  gets killed (exactly) by  $p_1$ , while  $f_1$  is a homology equivalence. (It is, of course, a homotopy equivalence already, but that is immaterial for our arguments.) Since  $C_1^2$  is far from  $R - r(1) = R(1) - b(1)$ , the following restriction of  $f_1$  is also a homology equivalence

$$\Gamma - R(1) = (\Gamma - b(1)) - (R(1) - b(1)) \xrightarrow{f_1} \Gamma \cup C_1^2 - \underbrace{(R(1) - b(1))}_{= R - r(1)};$$

this fact is graphically displayed in Figure 36, where  $R - r(1)$  is not at all represented; let us say that it is not physically present in the figure.

Since  $\Gamma - R$  is connected, so is also the  $\Gamma - (R - r(1))$ , and this implies that the following is connected too

$$(106) \quad \Gamma \cup C_1^2 - (R - r(1)) = \Gamma \cup C_1^2 - (R(1) - b(1)).$$

It follows that  $\Gamma - R(1)$  is connected. Next I CLAIM that  $H_1(\Gamma - R(1)) = 0$ .

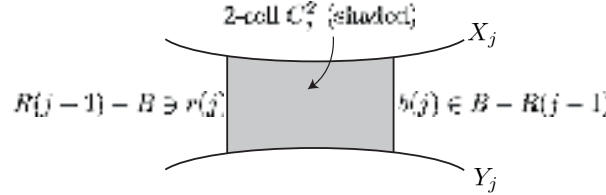
**[Proof of the Claim.** By going from  $\Gamma$  to  $\Gamma \cup C_1^2$ , we kill  $[r(1)] \in H_1(\Gamma) = H_1(\Gamma \text{ rel } (\Gamma - R))$ . When we further delete  $R(1) - b(1)$  in the RHS of the formula (106) and hence get the  $\Gamma - R(1)$ , we also kill all the other  $[x] \in H_1 \Gamma$ ; one uses here  $R(1) - b(1) = R - r(1)$ .] Hence  $R(1) \in P$ .

By now, the first step in our induction is completely taken care of.

For the step  $j - 1 \Rightarrow j$ , we replace Figure 36 by its higher analogue, Figure 37. In the context of this figure, we find that

$$(107) \quad \partial C_j^2 \cdot b(j) = 1 = \partial C_j^2 \cdot r(j) \quad \text{and} \quad C_j^2 \cdot (R(j-1) - r(j)) = 0 = \partial C_j^2 \cdot (R(j) - b(j)).$$

The argument used above can be extended to show that  $R(j) \in P$ . This clinches the inductive process, and we are left with proving point 5); the 4) should be clear already, by now.



**Figure 37.**

Illustration for (105.j), i.e. for  $\Gamma - R(j-1) = X_j \cup b(j) \cup Y_j$ , the higher stage of our intersection. Here

$$R(j-1) = R - \sum_{\ell=1}^{j-1} r(\ell) + \sum_1^{j-1} b(\ell) \supset \sum_{j+1}^{\infty} r(\ell).$$

This means that we have

$$\partial C_j^2 \cdot \sum_{j+1}^{\infty} r(\ell) = 0;$$

see the main text.

We have

$$R \xleftarrow[\text{inclusion}]{\Psi} B \cap R + \sum_1^{\infty} r(n) \xrightarrow[\text{bijection}]{\Phi} B,$$

and we need to prove that

$$T \equiv \Gamma - \left[ (B \cap R) + \sum_1^{\infty} r(n) \right]$$

is a tree (i.e. that  $H_0 T = Z$ ,  $H_1 T = 0$ ); in that case  $\Psi$  has to be bijective. [Of course, we could also have worked here with  $R \cap B$  deleted everywhere, i.e. with  $R - B \cap R \leftarrow \sum_1^{\infty} r(n) \rightarrow B - B \cap R$ , a.s.o.] And since,



anyway all our construction has proceeded with the  $B \cap R$  deleted, we certainly have  $\partial C_j^2 \cdot (R \cap B) = \emptyset$  and this allows us to introduce the space  $(\Gamma - B \cap R) \cup \sum_1^\infty C_j^2$ . Here, there are two maps, and in the formula below, the  $\ell$  means some arbitrary generic level:

$$T \xrightarrow{\zeta_1} (\Gamma - R \cap B) \cup \sum_1^\infty C_j^2 \xleftarrow{\zeta_2} (\Gamma - R \cap B) - \sum_{n=1}^\ell \ell(n) \cup \sum_{\ell+1}^\infty C_j^2.$$

For every  $j$ , we have  $\partial C_j^2 \cdot r(j) = 1$  and  $\partial C_j^2 \cdot \sum_{j+1}^\infty r(n) = 0$ , i.e. the matrix  $\partial C_j^2 \cdot r(i)$  is  $\text{id} + \text{nil}$  (of the easy type). This implies that, at the level of  $\Gamma$ , the following two operations: deleting  $\sum_1^\infty r(n)$  OR adding  $\sum_1^\infty C_j^2$  are, homologically speaking equivalent. This shows that  $\zeta_1$  is a homology equivalence.

Next, start with  $\sum_{n=1}^{j-1} b(n) \subset R(j-1) - r(j)$ , and then notice that, for all  $j$ 's,  $\partial C_j^2 \cdot b(j) = 1$  and  $\partial C_j^2 \cdot \sum_1^{j-1} b(n) = 0$ . [This last fact follows from

$$\begin{aligned} \partial C_j^2 &\subset \underbrace{\Gamma - R(j-1)}_{\subset \Gamma - \sum_1^{j-1} b(n)} + r(j). \end{aligned}$$

It follows that all the inclusion maps below are homology equivalences too

$$\left( \Gamma - R \cap B - \sum_1^\ell b(n) \right) \cup \sum_{\ell+1}^\infty C_j^2 \subset \left( \Gamma - B \cap R - \sum_1^{\ell-1} b(n) \right) \cup \sum_\ell^\infty C_j^2 \subset \dots \subset (\Gamma - B \cap R) \cup \sum_1^\infty C_j^2.$$

**[Explanation.** The inclusion

$$\left( \Gamma - R \cap B - \sum_{n=1}^\ell b(n) \right) \cup \sum_{\ell+1}^\infty C_j^2 \subset \left( \Gamma - R \cap B - \sum_{n=1}^{\ell-1} b(n) \right) \cup \sum_\ell^\infty C_j^2$$

consists in: i) leaving  $b(\ell)$  in place, undeleted, and ii) adding  $C_\ell^2$ . In the context of the map above,  $C_\ell^2$  goes once through  $b(\ell)$  and it also meets various  $b(L > \ell)$ . So it exactly cancels the increase of  $H_1(\dots)$  represented by leaving  $b(\ell)$  in place, undeleted.]

This implies that  $\zeta_2$  is a homology equivalence, just like  $\zeta_1$ .

Since  $\zeta_1$  is a homology equivalence and since we also have that  $(\Gamma - R \cap B) \cup \sum_1^\infty C_j^2$  is connected, the  $T$  is connected too. We have to show now that  $H_1 T = 0$ . So, we start with the inclusion  $T \xrightarrow{i} \Gamma$ . For any, arbitrary  $\ell$ , we have a commutative diagram where for the first equality ( $H_1 T = \dots$ ) we use the fact that both  $\zeta_1$  and  $\zeta_2$  are homology equivalences.

(108)

$$\begin{array}{ccc} H_1 T = H_1 \left( \left( \Gamma - R \cap B - \sum_1^\ell b(n) \right) \cup \sum_{\ell+1}^\infty C_j^2 \right) & \xrightarrow{i_*} & H_1 \Gamma = H_1(\Gamma \text{ rel } (\Gamma - B)) \\ & \searrow I_\ell \quad \swarrow \eta_\ell & \\ & H_1(\Gamma \text{ rel } (\Gamma - B)) / \left\{ \sum_1^\ell b(n), R \cap B, \sum_{\ell+1}^\infty C_j^2 \right\} & \end{array}$$

In the diagram above, the map  $i$  is an inclusion of graphs  $T \xhookrightarrow{i} \Gamma$ , making that, automatically,  $i_*$  is injective. Also, quite trivially  $\eta_\ell$  surjects and  $I_\ell$  is bijective.

**[Explanations.]** Start with the inclusion  $\Gamma = R \cap B - \sum_1^\ell b(n) \subset \Gamma$  and with the induced homology map  $H_1\left(\Gamma - R \cap B - \sum_1^\ell b(n)\right) \xrightarrow{\alpha_\ell} H_1(\Gamma) = H_1(\Gamma \text{ rel } \Gamma - B)$ . Here  $\text{Im } \alpha_\ell = H_1(\Gamma \text{ rel } \Gamma - B) / \left\{ R \cap B, \sum_1^\ell b(n) \right\}$ . When to  $\Gamma - R \cap B - \sum_1^\ell b(n)$  one adds  $\sum_{\ell+1}^\infty C_j^2$ , then at the level of  $H_1$  one gets the further quotient.

$$\text{Im } \alpha_\ell / \left[ \sum_{\ell+1}^\infty C_j^2, \text{ a.s.o.} \right]$$

Assume now that there exists some  $0 \neq x \in H_1 T$ . From (108) it follows that  $I_\ell(x) = \eta_\ell i_*(x) \neq 0$ . But then, when we kill

$$[R \cap B] + \sum_{j=1}^\ell [b(j)] + \sum_{\ell+1}^\infty C_j^2 \subset H_1(\Gamma \text{ rel } (\Gamma - B)),$$

for  $\ell \rightarrow \infty$ , all of  $H_1(\Gamma \text{ rel } (\Gamma - B))$  gets eventually killed. This means a contradiction. Hence  $H_1 T = 0$  and our lemma is proved.  $\square$

In all the little theory above,  $\Gamma$  was some generic, abstract graph. We move back now to our  $\Gamma(1) \subset \Gamma(\infty) = \Gamma(\infty) \times r \subset 2\Gamma(\infty)$  and to the two families  $R_0, B_0 \subset \Gamma(\infty) \times r = \Gamma(X^2[\text{new}])$  and the larger  $R_1, B_1 \subset 2\Gamma(\infty)$ , each of them four having the property  $P$  in their appropriate context. It is understood that the **subdivisions (like in Figure 30)**, are incorporated in these definitions. The  $\Gamma(1)$  is supposed to incorporate them too. The (93) will be understood in this extended context, with a new meaning for  $n, M$ . The  $\Gamma(1)$  is  $\Gamma(1) \times (\xi_0 = -1)$  and it has an excess of BLUE, in the sense that  $\Gamma(1) - B_0 \subset \Gamma(\infty) - B_0$  is not connected, in the tree  $\Gamma(\infty) - B_0$ ; it is actually violently disconnected, each vertex corresponds to a connected component. So,  $B_0 \cap \Gamma(1) \subset \Gamma(1)$  does not have property  $P$ .

I will show now a **preliminary construction** proceeding for the time being inside the  $\Gamma$  (actually our  $\Gamma(\infty)$ ) which is now fixed once and for all.

In our  $\Gamma(\infty) - B_0$  we can certainly find a geodetic arc  $g_1 \subset \Gamma(\infty) - B_0$  which joins two distinct components of  $\Gamma(1) - B_0$  and which only touches  $\Gamma(1) - B_0$  via its ends.

We can iterate this process inside  $\Gamma(\infty) - B_0$ , until we get

$$(109) \quad \Gamma(2) = \Gamma(1) \cup g_1 \cup g_2 \cup \dots \cup g_\chi \subset \Gamma, \quad \sum_1^\chi g_i \subset \Gamma - B_0,$$

which is now such that  $\Gamma(2) - B$  is a **tree**. In this situation, we will say that  $\Gamma(2)$  is **well-balanced** for BLUE. Also, the formula (109) **defines** for us the quantity  $\chi$ .

Now, the  $\Gamma(1)$  was well-balanced for RED, but the  $\Gamma(2)$  is NOT. We will express the BALANCING features just mentioned, by  $|\Gamma(1) \cap R_0| = b_1(\Gamma(1))$  (good balance for RED, the notation here being  $|\Gamma(1) \cap R_0| = \# \{ \text{of } R_0 \subset \Gamma(1) \}$ ). Next,  $|\Gamma(2) \cap B_0| = b_1(\Gamma(2))$ , but for  $\Gamma(2) - R_0 \subset \Gamma(\infty) - R_0$  we get a disconnected  $\Gamma(2) - R_0$ , coming with  $|\Gamma(2) \cap R_0| \geq b_1 \Gamma(2) = |\Gamma(2) \cap B_0|$ , hence (potentially at least), a disbalanced for RED.

In terms of the various Betti numbers, and with the quantity  $\chi$  **defined like in (109)**, (do not mix it up with the Euler characteristic), we have  $\chi = b_0(\Gamma(1) - B_0) - 1$ , by construction.

Then, as a direct consequence of the structure of (109)

$$\chi = b_1(\Gamma(2)) - b_1(\Gamma(1)) = \underbrace{|\Gamma(2) \cap B_0|}_{= \Gamma(1) \cap B_0} - \underbrace{|\Gamma(1) \cap R_0|}_{\not\subset \Gamma(2) \cap R_0}.$$

We will use the notations

$$R_0 \cap \Gamma(2) \supset R_0 \cap \Gamma(1) = \sum_{i=1}^n H_i^r, \quad R_0 - \sum_1^n H_i^r = \{h_1, h_2, \dots\},$$

where the order is chosen such that  $C \cdot h = \text{id} + \text{nilpotent}$ . Here  $C \equiv \Sigma \{C_i \text{ remaining}\} + \Sigma \{\text{little } C_i\}$ , see (101).

We will present now the real CONSTRUCTION which will realize the BALANCING of RED and BLUE, but this time 1-handles will slide and the ambient infinite graph  $\Gamma(\infty)$  (or  $\Gamma(2\infty)$ ) will be changed. To give an idea of what is going on, we start for illustration with the case when in (109) we have  $\chi = 1$  and also

$|\Gamma(2) \cap R_0| > b_1 \Gamma(2)$ . Our  $(\Gamma(1) - B_0) \cup g_1$  is now a tree and, since  $b_1 \left( \underbrace{\Gamma(1) \cup g_1}_{\text{now, this is } \Gamma(2)} - \sum_{i=1}^n H_i^r \right) > 0$ , the

$\Gamma(1) \cup g_1 - \sum_{i=1}^n H_i^r$  cannot be housed inside  $\Gamma(\infty) - R_0$ , and we find that  $g_1 \cap R_0 = \{x_1, x_2, \dots, x_p; y_1\} \neq \emptyset$ . The notation is chosen here such that, in the RED order  $y_1 > \{\text{the } x_i\text{'s}\}$ .

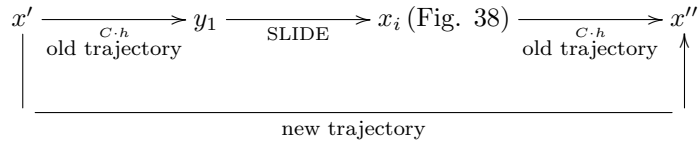
We slide now  $y_1$  over the  $x$ 's, as it is suggested to do in Figure 38. This defines a transformation  $\Gamma(\infty) \Rightarrow \Gamma(\infty)$  [balanced], where  $\chi = 1$ .

We have assumed here that  $|\Gamma(2) \cap R_0| > b_1 \Gamma(2)$  and, when it so happens that  $|\Gamma(2) \cap R_0| = b_1 \Gamma(2)$ , then  $\{y; x\} = \emptyset$  and our transformation is mute. Notice the following facts.

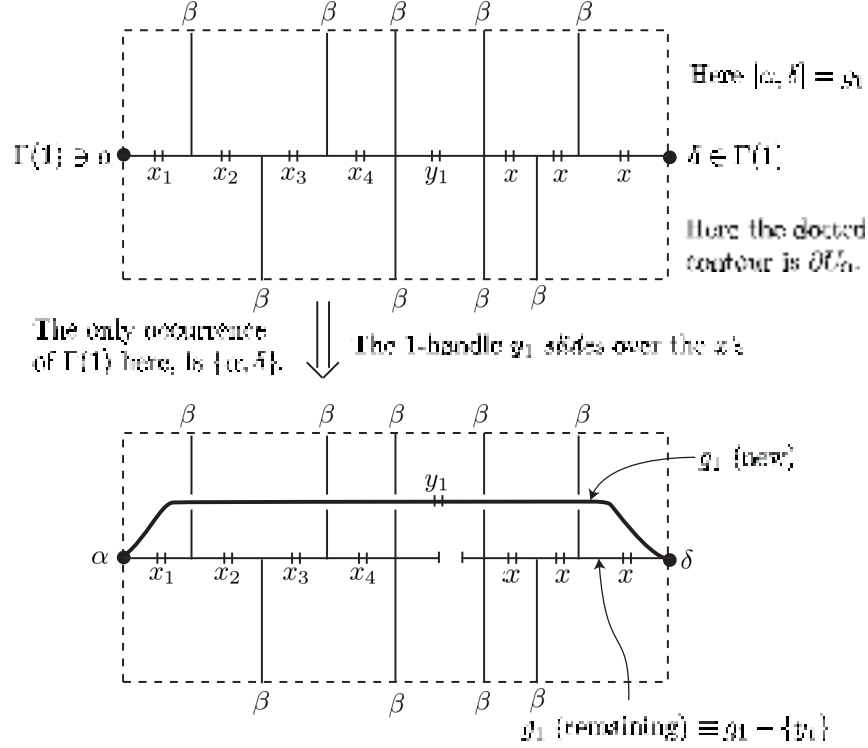
(110.1) Both  $\Gamma(\infty)$  [balanced]  $- R_0$  and  $\Gamma(\infty)$  [balanced]  $- B_0$  are now trees.

(110.2) In our special case ( $\chi = 1$ ), the  $\Gamma(1) - B_0$  has consisted of two trees; joined via  $g_1$ . It follows, that when we move from  $\Gamma(1) \subset \Gamma(\infty)$  to  $\Gamma(1) \cup g_1(\text{new}) \subset \Gamma(\infty)$  [balanced], then  $\Gamma(1) \cup g_1(\text{new})$  is now **well-balanced**, both for RED and for BLUE.

(110.3) Our transformation brings the following new trajectory for the RED oriented graph  $C \cdot h$ :



But this does not change the RED order of the  $h$ 's.



**Figure 38.**

Illustration for the transformation  $\Gamma(\infty) \Rightarrow \Gamma(\infty)$  [balanced]. In the upper figure, inside the dotted contour, we can see a neighbourhood  $g_1 = [\alpha, \delta] \subset U_0 \subset \Gamma(1)$ , which connects with the outer world through the sites  $\{\alpha, \delta, \beta\}$ . It is understood here that  $\{x, y_1\} \subset R_0 - B_0$ .

**The abstract balancing Lemma 15.**

1) We consider now the general case and introduce the quantity

$$(111) \quad \chi \equiv |\Gamma(2) \cap B_0| - |\Gamma(1) \cap R_0| = M - n,$$

already mentioned above.

The case  $\chi = 1$  has already been dealt with when we discussed Figure 38. In general, we have a disjoint partition into connected components, with  $\chi + 1 \geq 1$

$$\Gamma(1) - B_0 = \sum_{\alpha=1}^{\chi+1} X_\alpha \subset \Gamma(\infty) - B_0.$$

In  $\Gamma(\infty) - B_0$  there is a geodetic arc  $g_1$  connecting two connected components of  $\Gamma(1) - B_0$  and meeting  $\Gamma(1) - B_0$  only through its end-points. I CLAIM, at this point, that we have  $g_1 \cap R_0 \neq \emptyset$ .

Here is the **proof of this claim**. Assume that  $g_1 \cap R_0 = \emptyset$ . Then with  $(\Gamma(1) - R_0) \cup g_1$  hooked now between two trees

$$\Gamma(1) - R_0 \subset (\Gamma(1) - R_0) \cup g_1 \subset (\Gamma(\infty) - R_0),$$

and with the  $g_1$  which rests with both its ends on  $(\Gamma(1) - R_0)$ , we have  $b_1((\Gamma(1) - R_0) \cup g_1) > 0$ , which is a contradiction. The CLAIM is proved.

We have  $g_1 \cap R_0 = \{y_1; x_{11}, x_{12}, \dots, x_{1n}\}$ , with  $y_1 > \{x_{11}, \dots, x_{1n}\}$ , (the RED order is being meant here), and we can treat now the  $g_1$  like in the Figure 38. This leads to a FIRST TRANSFORMATION, where  $y_1$  slides over the  $\sum_i x_{1i}$ , call it

$$(112) \quad (\Gamma(\infty), \Gamma(1)) \implies (\Gamma(\infty)_1, \Gamma(1)_1 \equiv \Gamma(1) \cup g_1(\text{new})).$$

Here  $\Gamma(1)_1 - B_0$  has one connected component less than  $\Gamma(1) - B_0$ . So we have

$$\Gamma(1)_1 - B_0 = \sum_{\alpha=1}^x X_\alpha(1) \subset \Gamma(\infty)_1 - B_0.$$

There is now a next geodetic arc  $g_2 \subset \Gamma(\infty)_1 - B_0$  connecting, just like before two connected components of  $\Gamma(1)_1 - B_0$ , in a clean manner. Just like for (112) we have now

$$g_2 \cap R_0 = \{y_2; x_{21}, \dots, x_{2m}\}.$$

This means a transformation, analogous to (112)

$$(113) \quad (\Gamma(\infty)_1, \Gamma(1)_1) \implies (\Gamma(\infty)_2, \Gamma(1)_2 \equiv \Gamma(1)_1 \cup g_2(\text{new})).$$

After  $\chi$  steps we get the final composite transformation

$$(114) \quad (\Gamma(\infty), \Gamma(1)) \implies (\Gamma(\infty)[\text{balanced}], \Gamma(3) \equiv \Gamma(1)_\chi),$$

which is our BALANCING RED/BLUE CONSTRUCTION.

Our transformation

$$(\Gamma(\infty), \Gamma(1)) \implies (\Gamma(\infty)[\text{balanced}], \Gamma(3)),$$

which changes  $\Gamma(1)$  into  $\Gamma(3)$ , does not change the connections which  $\Gamma(1)$  had, already, with the outside world.

2) At the end of the construction above, both  $\Gamma(\infty)[\text{balanced}]$  and  $\Gamma(3)$  are well-balanced, both for BLUE and for RED. More explicitly, we find the following items

$$(115) \quad \begin{aligned} \Gamma(3) \cap B_0 &= \Gamma(1) \cap B_0 = \sum_1^M b(i) \text{ (see (93)) and} \\ \Gamma(3) \cap R_0 &\equiv \sum_1^M H_i^r \equiv \Gamma(1) \cap R_0 + \{\text{the } y\text{'s}\}, \text{ so that now} \\ b_1(\Gamma(3)) &= |\Gamma(3) \cap R_0| = |\Gamma(3) \cap B_0| = M. \end{aligned}$$

3) We consider  $B_0 \equiv \{b_1, b_2, \dots, b_M, b_{M+1}, \dots\}$  with  $\{b_1, b_2, \dots, b_M\}$  like above, written in the increasing BLUE order. I claim it is possible to order  $R_0$  starting with  $\Gamma(3) \cap R_0$ ,

$$R_0 = \{H_1^r, H_2^r, \dots, H_M^r; H_{M+1}^r, \dots\},$$

so that the following things should happen: one can apply the infinitistic Lemma 14 to  $R_0 \subset \Gamma(\infty)[\text{balanced}] \supset B_0$  in such a way that  $\Gamma(3)$  is invariant for the ABSTRACT COLOUR-CHANGING PROCESS, and that moreover

$$(115.1) \quad \Phi(H_i^r) = b_i \text{ for all } 1 \leq i \leq M.$$

**Proof.** The only item not already proved in the statement itself is the 3). So, since  $\Gamma(3)$  is already well-balanced for like RED and BLUE, we can start with a first application of Lemma 14 to the finite graph  $\Gamma(3)$ , so as to achieve just (115.1) for  $i \leq M$ . This fixes then an order on the  $\sum_1^M H_i^r$ . Next, we turn to our Lemma 14, this time in earnest, using its full infinite glory, and this time for the  $\Gamma(\infty)[\text{balanced}]$ , but without touching to  $\Gamma(3)$  any longer. Our ABSTRACT BALANCING LEMMA 15 is now proved.  $\square$

**Important Remark.** In the action of  $\Gamma(\infty) \Rightarrow \Gamma(\infty)[\text{balanced}]$ , Figure 38, the  $R_0 \cap B_0 \subset B_0$  gets deleted, from the very beginning. Hence one has not have to worry about  $R \cap B$ , neither in the context of the abstract Lemma 15 nor when it comes to its geometric implementation ( $t = 0$ )  $\Rightarrow$  ( $t = \frac{1}{2}$ ) explained below, leading to lemma 17.  $\square$

Retain, at this point, that by now all the four graphs are *trees*:

$$\Gamma(\infty)[\text{balanced}] - R, \Gamma(\infty)[\text{balanced}] - B, \Gamma(3) - R, \Gamma(3) - B.$$

Also our little abstract theory is now finished and the next section will show how to implement it, geometrically, in 4<sup>d</sup>.

Let us go back now to the pre-Lemma 15 family

$$R = \{R_1, R_2, \dots, R_n; h_1, h_2, \dots, h_p, \dots\} = (\text{with a different notation}) = \{\underbrace{H_1^r, H_2^r, \dots, H_n^r}_{\text{this is } R_1, R_2, \dots, R_n}; H_{n+1}^r, H_{n+2}^r, \dots\}.$$

In terms of notations from (8) and of the link (99.1), we have  $C \cdot h = \text{id} + \text{nil}$ , of easy type, establishing the duality  $\{C\} \approx \{h\}$ .

**Fact (116)** In terms of this last duality, these  $h$ 's which are dual to the {little  $C$ } 's are all in  $R \cap B$  and hence they can *never* occur among the  $\sum_1^{M-n} y_1^2$  from Lemma 15. The {little  $C$ } is here like in (97).

At this point, I will make a, for the time being *purely notational PROMOTION* for the 1-handles and the curves of our {link}, at a time before we start implementing geometrically the Lemmas 14, 15 and which I will call the time  $t = 0$ . So here is the

**Promotion Table:**

$$\sum_1^n H_i^r \Rightarrow \sum_1^M H_i^r \equiv \sum_1^n H_i^r + \sum_1^{M-n} y_j,$$

with the  $y_j$ 's promoted, honorifically at least, as 1-handles of  $\Delta_{\text{Schoenflies}}^4$ ; next

$$\sum_1^{\bar{n}} \Gamma_j \Rightarrow \sum_1^{\bar{n}} \Gamma_j \equiv \sum_1^{\bar{n}} \Gamma_j + \left\{ \text{the } C\text{'s dual to the } \sum y_j = \sum_{n+1}^M H_i^r \right\}.$$

Here the promotion of the  $C$ 's, as attaching zones of 2-handles of the  $\Delta_{\text{Schoenflies}}^4$  is really only *honorific*. The corresponding 2-handles are no longer directly to  $\Gamma(1)$  (with  $\Gamma(1) = \Gamma(1) \times (\xi_0 = -1)$ ), nor even to  $\Gamma(3)$ . The promotion will stay purely honorific, even when Lemma 15 will have been implemented geometrically. The next promotion are:

The mute promotion  $\sum_1^M b_i \xRightarrow{\approx} \sum_1^M b_i$ , where remember that our  $b_i$ 's are the  $(\Gamma(1) \times (\xi_0 = -1)) \cap B$ . And, after this trivially mute promotion, we have

$$\begin{aligned} \sum_1^\infty h_i &\implies \sum_1^\infty h_i - \sum_{n+1}^M H_j^r, \\ \sum_1^\infty C_i &\implies \sum_1^\infty C_i - \sum_{n+1}^M \Gamma_j. \end{aligned}$$

(Now  $\bar{n} \geq n, \bar{\bar{n}} = \bar{n} + (M - n)$ .)  $\square$

By definition, after this promotion we are at time  $t = 0$ , at the level of  $2X_0^2$  and we continue to have (at  $t = 0$ )

$$C \cdot h = \text{easy id} + \text{nilpotent}.$$

In the process leading to our final desired result in the next section VII, there will be other successive times after  $(t = 0)$ , let us say they are

$$t = 0, t = \varepsilon, t = 2\varepsilon, \dots, t = \frac{1}{2}, t = \frac{1}{2} + \varepsilon, \dots, t = 1.$$

With the new smaller families  $\sum_i C_i, \sum_k h_k$ , the new smaller

$$(117) \quad \text{LAVA}(t = 0) \equiv \sum_1^\infty h_j \cup D^2(C_j) \quad (\text{AFTER PROMOTION}),$$

continues to have the STRONG PRODUCT PROPERTY and, outside the sites living in the  $\sum_1^{\bar{\bar{n}}} \text{int } D^2(\Gamma_j)$ , the extended cocores continue to make sense.

We have  $\sum_1^{\bar{\bar{n}}} \Gamma_j \cap \text{LAVA}(t = 0) \neq \emptyset$ , since the promoted  $\Gamma_j$ 's can happily touch to LAVA  $(t = 0)$ . They are not directly attached to  $\Gamma(1)$  not to  $\Gamma(3)$  for that matter. But  $\left(\sum_1^{\bar{\bar{n}}} \Gamma_j\right) \cap \text{LAVA}(t = 0) = \emptyset$ , while generally speaking, we also have  $\sum_1^M H_i^r \cap \text{LAVA}(t = 0) \neq \emptyset$  and, using the LAVA  $(t = 0)$  one can build the extended cocore of the  $H_i^r$ 's ( $i \leq M$ ). When it so happens that  $H_i^r \cap \text{LAVA}(t = 0) = \emptyset$ , then we have extended cocores of  $H_i^r = H_i^r$  itself, or if one wants, the cocore  $H_i^r$  itself. This is a trivial case, on which we will not dwell.

Coming back to our successive times  $t$  from above, they parametrize the times of successive geometric constructions happening in  $4^d$ , inside  $N_1^4(2X_0^2)^\wedge$ , and they will be carefully described in the next section.

There will be a time  $t = \frac{1}{2}$ , when the  $R/B$ -BALANCING will have been geometrically realized and then a final time  $t = 1$  when the COLOURS WILL HAVE CHANGED sufficiently so as to make the following GRAND BLUE DIAGONALIZATION possible, with  $1 \leq i, j \leq M$

$$\boxed{\eta_j(\text{green}) \cdot \{\text{extended cocore } H_i^b\}^\wedge = \delta_{ij}}.$$

The point is that we will change the diffeomorphic model of  $\Delta_{\text{Schoenflies}}^4$ , more precisely of  $N^4(\Delta^2)$  so that the  $\{\text{extended cocore } H_{i \leq M}^b\}^\wedge$  will be its 1-handles, with the  $H_j^r$ 's completely out of the picture.

But then, besides  $t = \frac{1}{2}$  and  $t = 1$  there will be other intermediary times too. At any time, things like  $\sum_1^M H_i^r \cap \text{LAVA}(t \neq 0) \neq \emptyset$  will always be with us; they are unavoidable. With them, a RED diagonalization would conflict with the sacro-sancted CONFINEMENT. So, we will **never** try to diagonalize things like

$$\sum_1^M \eta_i(\text{green}) \cdot \sum_1^M H_j^r \text{ nor, of course } \sum_1^M \Gamma_i \cdot \sum_1^M H_j^r.$$

Finally, both the BLUE and the RED flow have to be used for our constructions. Both have to be controlled. Also, even after we will make sense geometrically of  $N^4(\Gamma(3))$  the  $\sum_1^M D^2(\Gamma_i)$  are still NOT directly attached to it and to make sense of the promoted things as handles of index one and two of  $\Delta^4$  we need to move to **an infinite set-up**. [Possibly a finite high truncation would do, but the infinite set up is, finally less messy and more manageable.]

Let us define, as a first 4<sup>d</sup> step

$$(118) \quad \mathcal{Z}^4(t=0) \equiv N^4(2\Gamma_1(\infty)) - \sum_1^\infty h_i \supset \sum_1^M H_i^r.$$

This can be easily compactified into

$$(119) \quad \hat{\mathcal{Z}}^4(t=0) \equiv \mathcal{Z}^4(t=0) \cup \varepsilon(\mathcal{Z}^4(t=0)) = \overline{\mathcal{Z}_0^4(t=0)} \subset N_1^4(2X_0^2)^\wedge.$$

**Easy fact (119.1)** The pair  $\left(\hat{\mathcal{Z}}^4(t=0), \sum_1^M H_i^r\right)$  is **standard**, i.e.

$$\left(\hat{\mathcal{Z}}^4(t=0), \sum_1^M H_i^r\right) \stackrel{\text{DIFF}}{=} \left(M \# (S^1 \times B^3), \sum_1^M (*) \times B^3\right).$$

**Lemma 16. (The time  $t = 0$  compactification)**

1) (Reminder) Modulo an appropriate, not necessarily ambient isotopy, inside our ambient space  $N_1^4(2X_0^2)^\wedge$ , for each  $\alpha$  (see (38)), we have

$$(120) \quad \begin{aligned} \text{LAVA}_\alpha(t=0)^\wedge &\equiv \{\delta \text{LAVA}_\alpha(t=0) \times [0, 1]\} \overbrace{\cup}^{\delta \text{LAVA}_\alpha(t=0) \times \{0\}} \{(\delta \text{LAVA}_\alpha(t=0) \times \{0\}) \cup \lambda_\alpha\} = \\ &= \overline{\text{LAVA}_\alpha(t=0)} \subset N_1^4(2X_0^2), \end{aligned}$$

where this last “=” is an equality of subsets inside  $N_1^4(2X_0^2)^\wedge$ .

More globally, we have

$$\overline{\text{LAVA}(t=0)} = \overline{\sum_\alpha \text{LAVA}_\alpha(t=0)} = \left(\bigcup_\alpha \text{LAVA}_\alpha(t=0)^\wedge\right) \cup \lambda_\infty \quad (\text{see (46)}).$$

2) The following object is a smooth compact 4<sup>d</sup> handlebody of genus  $M$

$$\hat{\mathcal{Z}}^4(t=0) \equiv \mathcal{Z}^4(t=0) \cup \bigcup_\alpha \text{LAVA}_\alpha(t=0) \cup \lambda_\infty,$$



and we have

$$(121) \quad \left( \hat{Z}^4(t=0), \sum_1^M \{ \text{extended cocore } H_i^r \}^\wedge \right) \stackrel{\text{DIFF}}{=} \left( M \# (S^1 \times B^3), \sum_1^M (*) \times B^3 \right).$$

3) There is a diffeomorphism for  $\Delta^4 = \Delta_{\text{Schoenflies}}^4$ , and of course, like usual, by  $\Delta^4$  we actually mean  $N^4(\Delta^2)$ ,

$$(122) \quad \Delta^4 \stackrel{\text{DIFF}}{=} \hat{Z}^4(t=0) + \sum_1^{\bar{n}} D^2(\Gamma_i) \subset N_1^4(2X_0^2)^\wedge \stackrel{\text{DIFF}}{=} \Delta^4 \cup \partial\Delta^4 \times [0, 1],$$

and this is how  $\Delta^4$  and (1) will be conceived from now on, at least as long as we stay at time  $t = 0$ . Via a not necessarily ambient isotopy in the ambient space  $N_1^4(2X_0^2)^\wedge$ , can be assumed that

$$\hat{Z}^4(t=0) + \sum_1^{\bar{n}} D^2(\Gamma_i) = \overline{N^4(2X_0^2)} = N^4(2X_0^2)^\wedge.$$

**Proof.** Using the product property of LAVA one gets a collapse

$$(122.1) \quad \hat{Z}^4(t=0) \xrightarrow{\pi} \hat{\mathcal{Z}}^4(t=0),$$

as follows. Start by collapsing away the

$$\sum_1^M \{ \text{extended cocore } H_i^r \}^\wedge \cap \left( \sum_\alpha \delta \text{LAVA}_\alpha \times [0, 1] \right),$$

and next collapse the  $\sum_\alpha \delta \text{LAVA}_\alpha \times [0, 1]$  too. This leads to a first diffeomorphism

$$\left( \hat{Z}^4(t=0), \sum_1^M \{ \text{extended cocore } H_i^r \}^\wedge \right) = \left( \hat{\mathcal{Z}}^4(t=0), \sum_1^M H_i^r \right),$$

which allows us to deduce our (121) from (119.1).

Next, we exhibit a big collapse

$$\hat{Z}^4(t=0) + \sum_1^{\bar{n}} D^2(\Gamma_i) \xrightarrow{\{\text{BIG } \pi\}} N^4(\Gamma(1)) + \sum_1^{\bar{n}} D^2(\Gamma_i),$$

which is all we need for (122). Here is now how to produce the Big  $\pi$ .

After our PROMOTION, we have for our  $\Delta_{\text{Schoenflies}}^4$  the 1-handles  $H_i^r$ , with  $i = 1, 2, \dots, n, n+1, \dots, M$  and attaching curves  $\Gamma_j$ , with  $j = 1, 2, \dots, \bar{n}, \bar{n}+1, \dots, \bar{\bar{n}}$ , when  $\bar{\bar{n}} - \bar{n} = M - n$ . All this is purely honorific, abstract story, of course. We may assume here, concerning the ordering of the  $H_i^r$ 's, that, in the RED order, if  $n < j < i \leq M$ , then  $H_i^r > H_j^r$ , so that

$$(123) \quad \Gamma_{\bar{n}+i} \cdot \{ \text{extended cocore } H_j^r \} = 0 \quad \text{and} \quad \Gamma_{\bar{n}+i} \cdot \{ \text{extended cocore } H_i^r \} = 1.$$

Remember, also, that  $M - n = \bar{\bar{n}} - \bar{n}$ .

The increasing RED order means here that in the oriented graph gotten from the off-diagonal part of the RED geometric intersection matrix, the arrows go like this:  $H_M^r \longrightarrow H_{M-1}^r \longrightarrow \dots \longrightarrow H_{n+1}^r$ .

To get our  $\{\text{BIG } \pi\}$  we proceed via the following steps.

a) The (123) allows us to collapse away, first LAVA ( $t = 0$ )  $\cap \{\text{extended cocore } H_M^r\}$ , next  $H_M^r \cup D^2(\Gamma_M)$ .

b) Next, using again (123) we can proceed similarly, in order, for  $H_{M-1}^r, H_{M-2}^r, \dots, H_{n+1}^r$ . All these collapses are compatible with the  $\pi$  (122.1) and, since we know already that  $\sum_1^{\bar{n}} \Gamma_i \cap \text{LAVA} = \emptyset$ , all the rest of LAVA ( $t = 0$ ) can be collapsed away too, leaving us with a naked, i.e. lava-free, last collapse

$$\left( Z^4(t=0) - \sum_{n+1}^M H_i^r \right) \cup \sum_1^{\bar{n}} D^2(\Gamma_i) \searrow N^4(\Gamma(1)) \cup \sum_1^{\bar{n}} D^2(\Gamma_i).$$

This ends the Proof of Lemma 16.  $\square$

For the  $\sum_1^M b_i = B \cap \Gamma(1)$ , with the indexing from (115.1), we will also use from now on the notation  $\sum_1^M H_i^b$ . And here, just like for  $\sum_1^M H_i^r$ , we have the

$$(*) \quad \sum_1^M \{\text{extended } H_i^b\}^\wedge \subset \hat{Z}^4(t=0),$$

defined via the RED flow  $C \cdot h$ .

Eventually it will be  $(*)$  which will be the system of 1-handles of our  $\Delta^4$  (122).

We also have now our system of embedded exterior discs from Theorem 13, i.e. the

$$(124) \quad \sum_1^M (\delta_i^2, \eta_i(\text{green}) \equiv \partial \delta_i^2) \xrightarrow{\text{embedding } J} (\partial N^4(2X_0^2)^\wedge \times [0, 1] \subset N_1^4(2X_0^2)^\wedge \text{ (our ambient space), } \partial N^4(2X_0^2)^\wedge).$$

In lieu of the simple-minded  $\eta_i(\text{green}) \cdot b_j$ , which was diagonal, we consider now the more complex geometric intersection matrix

$$(125) \quad \eta_i(\text{green}) \cdot \{\text{extended cocore } H_j^b\}^\wedge = \delta_{ij} + \{\text{a } \textbf{parasitical} \text{ off-diagonal term } \eta_i(\text{green}) \cdot h_k, \text{ where } h_k \in R_0 - B_0 \text{ and, also } h_k \subset \{\text{extended cocore } H_j^b\}\}. \text{ A finite set of } k\text{'s is involved here.}$$

Here  $1 \leq i, j \leq M$ . Moreover, it is the little blue diagonalization from (101.1), in its full glory, which makes that, in the context of (125) we have that the parasitical  $h_k$ 's are  $h_k \in R_0 - B_0 = R_1 - B_1$ .

The presence of these parasitical  $h_k \in \eta_i(\text{green})$ ,  $i \leq M$ , mean that, at this stage in the game, the exterior discs

$$\sum_1^M (\delta_i^2, \partial \delta_i^2 = \eta_i(\text{green}))$$

are certainly NOT in cancelling position with the 1-handle  $\sum_1^M \{\text{extended cocore } H_i^b\}^\wedge$ .

The aim of the next section is to exhibit a handlebody decomposition for our  $N^4(\Delta^2)$  where the  $\sum_1^M \{\text{extended cocore } H_i^b\}^\wedge$  **are** the 1-handles.

And then, via the COLOUR-CHANGE, we will put the  $\sum_1^M \delta_i^2$  in cancelling position with them. This will allow us, eventually, to appeal to Lemma 3 and show that  $\Delta_{\text{Schoenflies}}^4 \in \text{GSC}$ .

The aim stated above will be achieved by realizing geometrically in  $4^d$  the ABSTRACT LEMMAS 14 and 15.

## 7 Balancing and change of colour, done now geometrically (The $4^d$ geometry of the passage $(t=0) \Rightarrow (t=1/2) \Rightarrow (t=1)$ and the compactifications at times $t=\frac{1}{2}$ and $t=1$ .)

We consider now Lemma 15 and its proof and we will concentrate on the typical step (112) which, keeping in mind the notations from the Figure 38, we write now as

$$(126) \quad (\Gamma(2\infty), \Gamma(1)) \Rightarrow (\Gamma(2\infty)_1[\text{balanced}], \Gamma(1) \cup g_1(\text{new})).$$

This is the first of the  $\chi$  steps to the  $\Gamma(2\infty)[\text{balanced}]$  in (114) and here  $g_1 \cap R_0 = \{y; x_1, x_2, \dots, x_n\}$ , with

$$(126\text{-bis}) \quad y_1 = H_{n+1}^r > \{x_{11}, \dots, x_{1m}\} \text{ (in the RED order of } 2X_0^2\text{); the } y_1 \text{ is PROMOTED and, after this promotion } \{x_{11}, \dots, x_{1m}\} \subset \sum_1^\infty h_i.$$

We want to realize now this step (126), geometrically inside  $N_1^4(2X_0^4)^\wedge$ , as

$$(126\text{-ter}) \quad (N^4(2\Gamma(\infty)), N^4(\Gamma(1))) \Rightarrow (N^4(2\Gamma(\infty))_1[\text{balanced}], N^4(\Gamma(1) \cup g_1(\text{new}))).$$

So now, in real life, and not just abstractly as in the last section, an **active** 1-handle  $y(\text{non-LAVA}) = (B_a^3 \times [-\varepsilon, \varepsilon], \partial B_a^3 \times [-\varepsilon, \varepsilon]) \subset (N^4(2\Gamma(\infty)), \partial N^4(2\Gamma(\infty)))$  is sliding over a passive 1-handle  $x(\text{LAVA}) = (B_p^3 \times [-\varepsilon, \varepsilon], \partial B_p^3 \times [-\varepsilon, \varepsilon]) \subset (N^4, \partial N^4)$ . Really very schematically, what we talk about here is suggested in Figure 39.

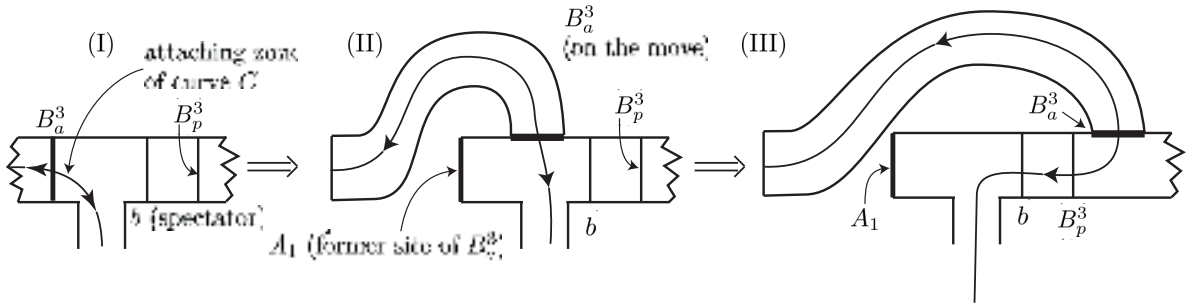


Figure 39.

A highly schematical  $2^d$  representation of the sliding of an **active** handle, with  $B_a^3$  as cocore, over another RED handle, the  $B_p^3$  (**passive**). Besides the RED 1-handles  $B_a^3, B_p^3$ , there is here also a **spectator** BLUE 1-handle  $b$ . The scenario for our step is  $I \Rightarrow II \Rightarrow III$ .

LEGEND:  $\leftrightarrow$  = this suggests the attaching zone of a 2-handle, resting on  $B_a^3$ , which gets dragged via covering isotopy. It is painted green.

But a much more detailed description than the one provided by Figure 39 is actually necessary.

(126.1) We want to preserve the STRONG PRODUCT property of LAVA, and, during our step (125) the  $C \cdot h = (\text{easy}) \text{ id} + \text{nil}$  might get violated. In order to take care of this we will need to add to our LAVA both *LAVA bridges* and *LAVA dilatations*.

[**A philosophical comment.** In this paper, there are some sacro-sancted things never to be violated, while others may be.

Here are our two sacro-santed principles which can never be violated nor trespassed:

I) The CONFINEMENT condition, inside  $\partial N_+^4(2\Gamma(\infty))$ .

II) The STRONG PRODUCT PROPERTY of LAVA.

Now when LAVA was first introduced, II) was assured by the condition  $C \cdot h = \text{id} + \text{nilpotent}$ . But then, when 1-handles will start sliding over each other, dragging the 2-handles along, then

III) The property  $C \cdot h = \text{id} + \text{nilpotent}$  *might get violated*.

The II) will be maintained by internal LAVA operations. Also, the II) is enough for defining the  $\{\text{extended cocore } x\}^\wedge$ , which will be with us, even with the violation from III.)

The violation of  $C \cdot h = \text{id} + \text{nil}$  will actually come later. Since right now  $y > x$  (RED order) there is yet not danger.

Remembering that LAVA is growing out of

$$\delta \text{LAVA} \subset \partial \left( N^4(2\Gamma(\infty)) - \sum_1^\infty h_i \right) \equiv \partial M^4,$$

and for building up a LAVA bridge, we start by adding, inside  $\partial M^4$  a  $3^{\text{d}}$  1-handle  $\delta \mathcal{H} \subset \partial M^4$  to  $\delta \text{LAVA}$ , and then we let it grow into a  $\delta \mathcal{H} \times [0, 1]$ , glued in an obvious way to the rest of LAVA. The LAVA dilatations are defined similarly.

(126.2) We also want to respect the basic SPLITTING  $\partial N^4(2\Gamma(\infty)) = \partial N_-^4(\Gamma_1(\infty)) \underbrace{\cup}_{\Sigma_\infty^2} \partial N_+^4(2\Gamma(\infty))$

(even with the change of  $2\Gamma(\infty)$  in (126)). And, with this, the forced CONFINEMENT conditions (101.1-bis) should be respected too.

At all times in the process (I)  $\Rightarrow$  (II)  $\Rightarrow$  (III), Figure 39,  $B_{(a \text{ or } p)}^3$  should be SPLIT as

$$B^3 = \frac{1}{2} B^3(+) \underbrace{\cup}_{\sigma^2 = \Sigma_\infty^2 \cap B^3 (= 2\text{-cell})} \frac{1}{2} B^3(-).$$

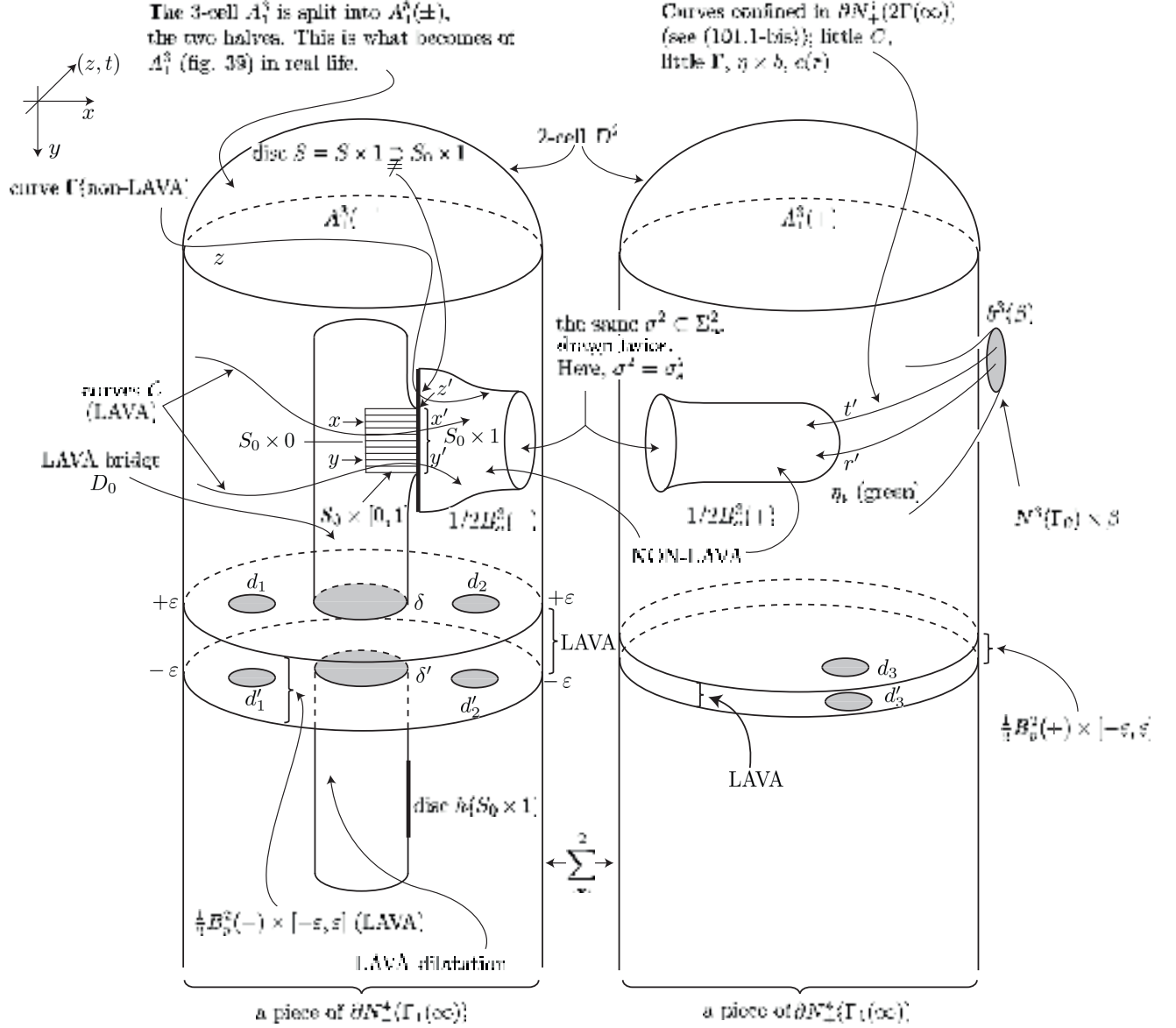


Figure 40.

A realistic view of Figure 39-(II). Separated into two distinct  $\pm$  halves, this figure suggests two  $3^d$  pieces out of which a small part of the bigger  $\partial N^4(2\Gamma(\infty))$  can be reconstructed. For typographical reasons, several geometric ingredients occur twice in this figure. This is, for instance, the case for  $D^2, \sigma^2, \Sigma_\infty^2$ .

We also have here  $S_0 \times [0, 1] \subset \delta \text{LAVA}$ , with  $S_0$  a disc. The LAVA just mentioned is actually LAVA *dilatation*.

We can reconstruct here also  $\partial B_p^3 = \frac{1}{2} B_p^2(-) \cup \frac{1}{2} B_p^2(+)$  and the whole of  $B_p^3 \times [-\varepsilon, \varepsilon]$  is LAVA. Of course,  $\text{int } B_p^3$ , living inside  $N^4(2\Gamma(\infty))$  is not visible here. But more LAVA is present here

than what is drawn: through the pairs of shaded discs  $d_1 + d'_1$ ,  $d_2 + d'_2$ ,  $d_3 + d'_3$  go attaching zones  $C_i \times D_i^*$  of 2-handles belonging to LAVA. These things are formally 1-dimensional, HENCE THEY ARE NOT IN THE WAY for the  $B_a^3$  moving from its initial position where it is glued to LAVA (with bridges and dilatation included, along  $\text{LAVA} \supset S_0 \times 1 \subsetneq S \times 1 \subset \partial B_p^3$ .) The isotopy

$$\begin{array}{ccc} S_0 \times 1 & \Longrightarrow & h(S_0 \times 1) \\ \downarrow & & \downarrow \\ S \times 1 & \Longrightarrow & h(S \times 1) \end{array}$$

happens along an arc we call  $\lambda$ , which is essentially our **long cigar**

$$(\text{LAVA bridge}) + \text{lava dilatation} \subset \partial N_-^4(2\Gamma(\infty)).$$

We have  $\lambda \cap (C_i \times D_i^*) = \emptyset$ .

#### Further comments and explanations concerning the Figure 40.

At the points  $x', y'$ , and  $t', v'$ , the various curves (attaching zones of 2-handles) climb on the moving 1-handle, as the arrows indicate.

As an immediate consequence of the inequality in (126-bis) we have that

$$(126.3) \quad \{\text{extended cocore } B_a^3\} \cap B_p^3 = \emptyset.$$

Here is the  $\delta$  LAVA which is explicitly present in Figure 40:

$$\text{The pieces } \{D_0 = \delta \text{ LAVA bridge}\} + \{\delta \text{ LAVA dilatation}\} + \frac{1}{2} B_p^3(\pm) \times \{-\varepsilon, +\varepsilon\}.$$

These three pieces are glued together via  $\delta, \delta'$ . Moreover, we have

$$S_p^2 \times (-\varepsilon, +\varepsilon) \subset \partial \text{LAVA} - \delta \text{LAVA}.$$

We have, also

$$S_p^2 = \partial B_p^3 = \partial \left( \frac{1}{2} B_p^3(+) \cup_{\sigma^2} \frac{1}{2} B_p^3(-) \right) = \frac{1}{2} B_p^2(+) \cup \frac{1}{2} B_p^2(-).$$

The  $B_p^3 \times [-\varepsilon, \varepsilon]$  is hidden from our sight, in the fourth dimension, and  $B_p^3 \times \{-\varepsilon, +\varepsilon\} \subset \delta \text{LAVA}$ . BUT, things like  $d_i(\text{shaded}) \times [-\varepsilon, +\varepsilon]$ , groing from  $d_i$  to  $d'_i$  are in  $\partial \text{LAVA} - \delta \text{LAVA}$ . Here, some  $D^2(C)(\text{LAVA})$  gets attached to  $N^4(2\Gamma(\infty))$ . Same for  $\delta \times [-\varepsilon, +\varepsilon] \subset \partial \text{LAVA} - \delta \text{LAVA}$ .

Inside the  $\frac{1}{2} B_p^2(-) \times [-\varepsilon, \varepsilon]$  our cigar melts into LAVA and disappears from the visual field.

The  $A_1^3 = A_1^3(-) \cup_{D^2} A_1^3(+)$  from Figure 40 corresponds to the site  $A_1$  visible in the Figures 39-(II, III).

It is the former position of  $B_a^3$ , before the sliding starts (see Figure 39-(I) too).

(126.4) Very importantly, there is no  $B$ , certainly no  $B \cap R$ , present in the context of the step from Figure 40. This fact stems from the following item: in the context of the Abstract Lemma 15, when it comes to the arcs  $\sum_1^x g_i$  which houses the  $\{y; x_1, x_2, \dots, x_n\}$ 's, we automatically have

$$\left( \sum_1^x g_i \right) \cap B = \emptyset.$$

End of (126.4).

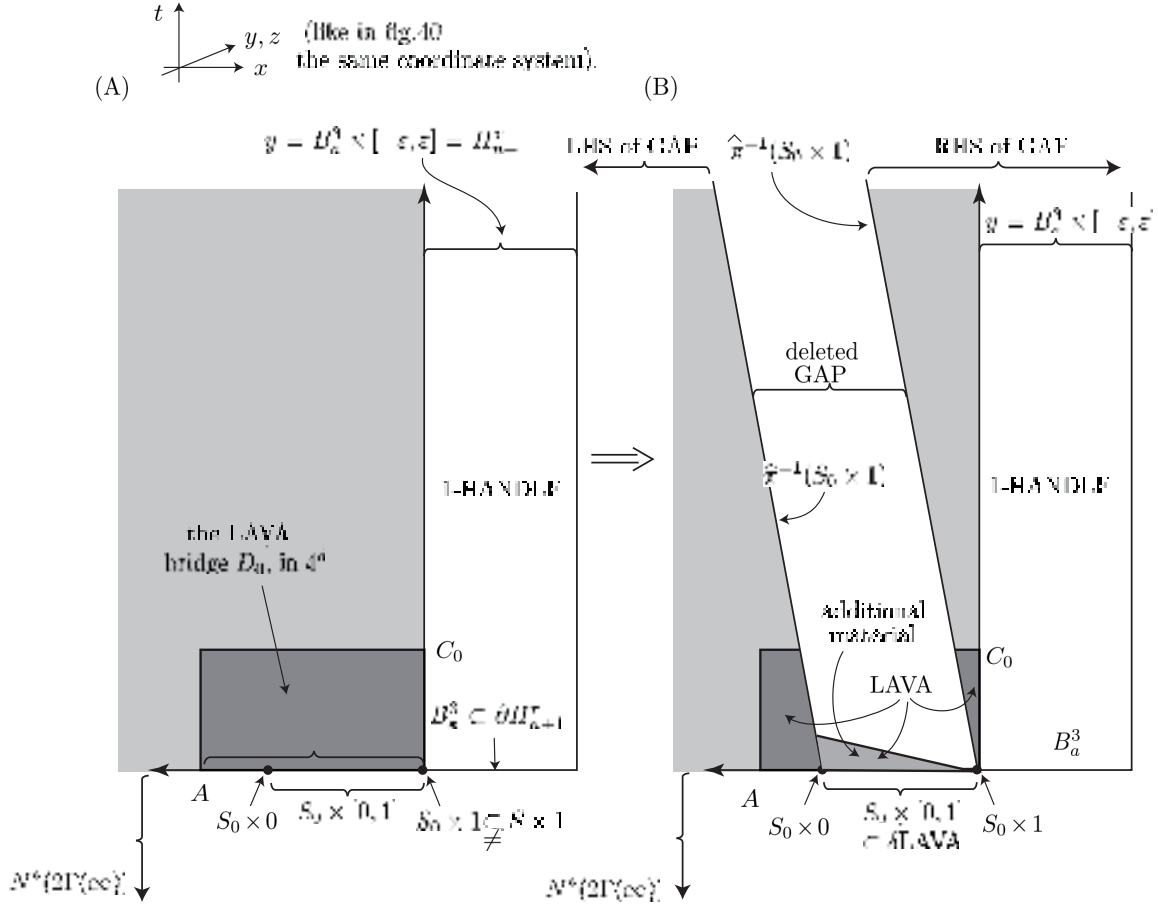


Figure 41.

Figure (A) is drawn at the level of Figure 40 which here corresponds just to the line on the bottom (= the bottom floor). The  $S_0 \times [0, 1]$  here is the one in Figure 40 and the doubly shaded area corresponds to the LAVA bridge  $D_0$ . We are supposed to be here in the plane  $(x, t)$  of one of the  $D^2(C)$ , like the  $C$  going through  $[x, x']$  in Figure 40. So, everything shaded (simply or doubly, is here LAVA. The  $[A, S_0 \times 0, S_0 \times 1]$  corresponds to the  $\delta$  LAVA(bridge), and it is located at the bottom floor. The (B) will be explained later, in the main text. And the  $S \times 1$  spreads in the  $(y, z)$  direction, surrounding  $S_0 \times 1$ .

LEGEND:  $\blacksquare$  = LAVA dilatation, in  $4^d$ ,  $\square$  =  $D^2(C)$ ,  $\leftrightarrow$  =  $C$  (attaching curve of the 2-handle  $D^2(C)$ ). The  $[ABC_0]$ -line is part of it. It CLIMBS OVER  $D_0$ ; in Figure 40 we see only the projection of this on the **floor**  $\partial N^4(2\Gamma(\infty))$  of the present figure.

Finally, in our Figure 40, enough things have been added to LAVA so that, in its move  $S \times 1 \subset \partial B_p^3$  should stay glued to it, via the  $S_0 \times 1 \subset S \times 1$ . The reason why  $\Gamma_j$  is treated differently than the  $C_i$ 's, in Figure 40 is that, for it, the kind of extended cocore producing the GAP in Figure 41-(B) is not available to us.

End of the comments and explanations concerning Figure 40.

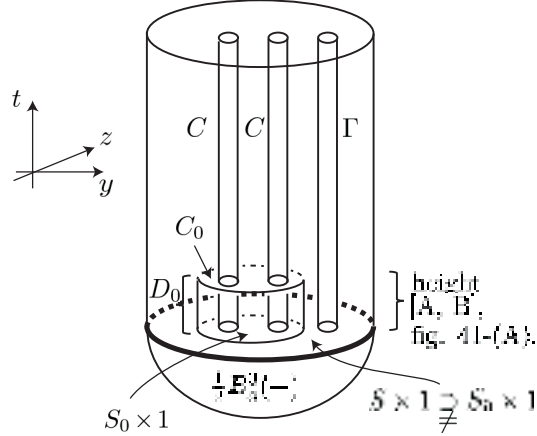
BUT there is also the following big problem with our elementary step, as in Figure 40 may suggest it, namely the following item:

(127) In the move along the isotopy  $S \times 1 \Rightarrow h(S \times 1)$ , the action handle  $B_a^3$  certainly stays glued to LAVA (this is the purpose of the LAVA bodies). BUT, at some point in this move, the  $S \times 1 \subset \partial B_a^3$  has to leave the  $\partial(\delta \text{LAVA})$ , and cover a piece of the naked lateral surface of the passive 1-handle  $B_p^3 \times [-\varepsilon, \varepsilon]$ , i.e. a piece of

$$\partial B_p^3 \times [-\varepsilon, \varepsilon] \subset \partial \text{LAVA} - \delta \text{LAVA}.$$

End of (127).

This is a potential danger for our sacro-sancted STRONG PRODUCT PROPERTY of LAVA. In what follows next we will show how to circumvent this potential difficulty.



**Figure 42.**

A section  $x = \text{const}$ ; here  $x = x(S_0 \times 1)$ , in terms of Figure 41-(A). We see the lateral surface of the 1-handles  $H_{n+1}^r = B_a^3 \times [-\varepsilon, \varepsilon]$ , Figure 41-(A) and the lateral surface of the cigar  $D_0$ , glued to each other. The  $((\text{curve } C) \cap D_0)$  is here an  $x$ -projection. The  $\Gamma$ , unlike the  $C$ , does not ride on the LAVA bridge  $D_0$ .





As said above, the slide of  $y = B_a^3$  over the  $x = B_p^3$  (corresponding to  $S \times 1 \Rightarrow h(S \times 1)$ ), if conducted in the obvious naïve manner (which Figure 40 may suggest), is not clearly preserving the product property of LAVA. So we will present another transformation, going from the same initial  $S \times 1$  to the same final  $h(S \times 1)$ , but in a manner where, in a manifest way, the PRODUCT PROPERTY is preserved at *every* intermediary moment. And, as far as we are concerned, it is only the final stage, at  $t = 1/2$ , which interests us, and not the details of  $(t = 0) \Rightarrow (t = \frac{1}{2})$ , when the PRODUCT PROPERTY OF LAVA ( $t = \frac{1}{2}$ ) is concerned.

$$(128) \quad \begin{array}{ccc} \text{LAVA}^\wedge & \xrightarrow{\hat{\pi}} & \delta \text{ LAVA} \\ \uparrow & & \updownarrow \text{id} \\ \text{LAVA} & \xrightarrow{\pi} & \delta \text{ LAVA}, \end{array}$$
$$\delta L_0^4 \equiv \delta\{\text{LAVA}(t=0) + \text{bridges and dilatations}\} \supset S_0 \times [0, 1],$$

What follows next, starting with Figure 41-(B) will be a purely internal LAVA affair, concerning  $L_0^4$ , not reflected outside of LAVA, in particular not reflected for the time being on the  $D^2(\Gamma)$  (see Figure 40, where both  $C$ 's and  $\Gamma$ 's climb on  $B_a^3 \times [-\varepsilon, \varepsilon] = y$ ) and  $D^2(\Gamma)$  will be relocated only in the end. With this, we introduce the following object:

$$(129) \quad \text{GAP} \equiv \hat{\pi}^{-1}(S_0 \times [0, 1]) \subset \hat{L}_0^4, \quad \text{see Figure 41-(B),}$$

which is split from the rest of  $\hat{L}_0^4$  by

$$(130) \quad \Sigma \equiv \hat{\pi}^{-1}(S_0 \times 0) + \hat{\pi}^{-1}(S_1 \times 1),$$

and we will also consider

$$(131) \quad \Sigma_1 = \Sigma \cup (S_0 \times [0, 1]) \subset \partial \text{GAP},$$

separating the GAP from the rest of the world. All this little story above can be vizualized in the Figures 41-(B) and 43. [But then, at the level of Figure 40, where it was introduced, our  $S_0 \times [0, 1]$  was a part of the  $3^d$   $\delta$  (LAVA bridge) and it looks that, when we delete the GAP, as we will do, it will totally disappear. We do not want this to happen, and so in Figure 41-(B) we have left a bit of  $4^d$  **{additional material}**  $\subset$  LAVA bridge (shaded and shielding  $S_0 \times [0, 1]$ ), which should stay put (i.e. undeleted) when the GAP is taken away. But we will not stress this in our notations.] As said, these things are vizualizable in Figures 41-(B), 43. The point is here the following

(132) If we consider the following **closed** subsets of LAVA  $(t = 0)^\wedge \equiv \hat{L}_0^4$ , namely  $\hat{\Lambda}^4 \equiv \{\hat{L}_0^4$  with that GAP deleted, modulo the splitting hypersurface  $\Sigma\} \subset \hat{\Lambda}_1^4 \equiv \hat{\Lambda}^4 \cup (S_0 \times [0, 1]) \cup \{\text{the additional material, shaded in Figure 41-(B)}\}$  and resting on  $S_0 \times [0, 1]$ , where the  $\hat{\Lambda}^4$  is a manifold and  $\hat{\Lambda}_1^4$  not (even if we leave in place the  $4^d$  additional material just mentioned) then, at the **local** level of  $S_0 \times 1$ , in Figure 41-(B), the two pieces of  $\hat{\Lambda}_1^4$ , namely the LHS of the  $\text{GAP} \supset S_0 \times 0$  and the RHS of the  $\text{GAP} \supset$  the 1-handle  $y$ , **only** hang together, at level  $\hat{\Lambda}_1^4$ , via  $S_0 \times 1$ , which is codimension two. End of (132).

Of course  $\hat{\Lambda}_1^4$  is connected, globally, and  $\dim(S_0 \times 1) = 2$ , as said. As a reflex of (128), we have two PROPER Whitehead collapses, both infinite of course,  $\hat{\pi}(\hat{\Lambda}^4)$  and  $\hat{\pi}(\text{GAP})$ , occurring in the formulae below

$$(132.1) \quad \hat{L}_0^4 \xrightarrow[\text{DELETION}]{} \Lambda^4 \xrightarrow{\hat{\pi}(\hat{\Lambda}^4)} (\delta \text{LAVA}) \cap \hat{\Lambda}^4,$$

and

$$(132.2) \quad \text{GAP} \xrightarrow{\hat{\pi}(\text{GAP})} \Sigma_1$$

out of which we can reconstruct  $\hat{\pi}$  (of course  $\hat{L}_0^4 = \Lambda^4 \cup \text{GAP}$ ), here just like  $\text{LAVA}^\wedge$  in (128), the  $\hat{L}_0^4$  also has the PRODUCT PROPERTY, expressed via the  $\hat{\pi}(\hat{L}_0^4)$ .

We can perform now in order, the following steps

(132.3) a) Extend the first line in (132.1) to the infinite collapse

$$\{\hat{\Lambda}_1^4 \text{ without the } (\{\text{additional material}\} - S_0 \times [0, 1])\} \xrightarrow{\hat{\pi}(\hat{\Lambda}_1^4)} \delta \text{LAVA}.$$

b) Next, perform the compact dilatation

$$\{\hat{\Lambda}_1^4 \text{ without the } (\{\text{additional material}\} - S_0 \times [0, 1])\} \longrightarrow \Lambda_1^4 \text{ (132) (which includes the additional material).}$$

The two items a), b) together, express the PRODUCT PROPERTY (albeit a singular one) for  $\hat{\Lambda}_1^4$ . This is, schematically,

$$\hat{\Lambda}_1^4 \xrightarrow{\hat{\pi}(\hat{\Lambda}_1^4)} \delta \text{LAVA}.$$

c) Finally, refine the (132.2) to

$$\text{GAP} \xrightarrow{\hat{\pi}(\text{GAP})} \{\Sigma_1 \text{ REDEFINED, by changing its bottom } S_0 \times [0, 1] \text{ into the roof of the additional material, resting on } S_0 \times [0, 1], \text{ which is shaded in the Figure 41-(B)}\}.$$

With these things, I claim now that the pair

$$\left( [\hat{\Lambda}_1^4 \cup \{\text{additional material (Fig. 41-(B))}\}] \underbrace{\quad}_{\Sigma_1 \text{ (REDEFINED)}} \text{GAP}, \delta L_0^4 \right),$$

has the PRODUCT PROPERTY.

All these various last items are the ingredients to be used in the next moves (133.I), (133.II). End of (132.3)

For the little story above, one should keep in mind that

$$\delta \text{LAVA} = [(\delta \text{LAVA}) \cap \Lambda^4] \cup [S_0 \times [0, 1]].$$

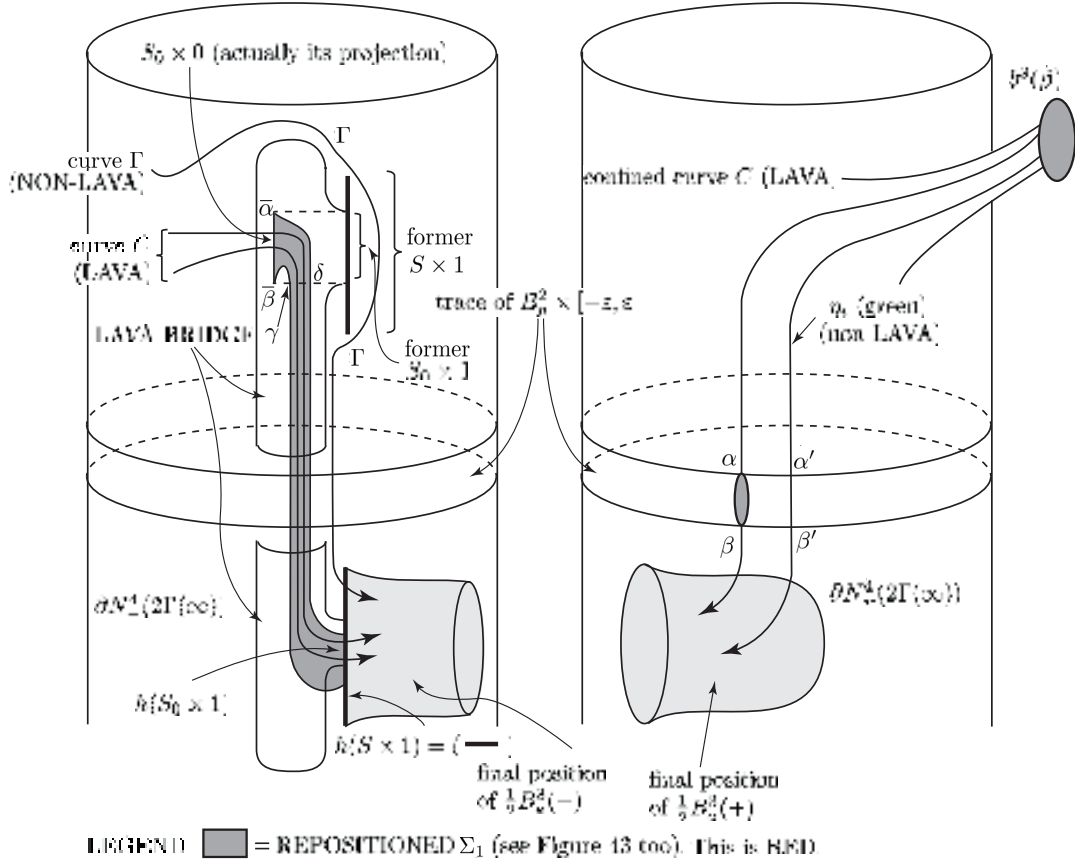


Figure 44.

#### Explanations for the Figure 44.

This figure, which should be compared to 40, suggests the effect of the step (133.I), on the site of the Figure 40. Many obvious notations, which should be just like in the Figure 40 have not been reproduced again here. The strongly shaded area is part of the REPOSITIONED  $\Sigma_1$ . We have

$$\{\text{final } B_a^3\} \supset h(S \times 1) \supsetneq h(S_0 \times 1) \subset \text{LAVA}.$$

The REPOSITIONING brings the

$$\{\text{RHS of the GAP, Fig. 41-(B)}\} \supset \{\text{the active 1-handle } B_a^3 \times [-\varepsilon, +\varepsilon]\},$$

into a correct position, so that it is glued now to  $h(\delta \times 1)$ .

Here, the  $\Sigma_1$  is first REDEFINED like in (132.3) and next REPOSITIONED, like above.

End of the explanations.

With these things, there is now an alternative procedure for performing our sliding of  $y = B_a^3$  over  $x = B_p^3$  in a way in which the initial and final stages are exactly as before, but where the PRODUCT property is manifest at all intermediary stages.

(133.I) Taking advantage of (132) we relocate on top of  $N^4(2\Gamma(\infty)) - B_a^3(y)$ , in the presence of  $D_0$ , and CORRECTLY, the

$$\{\text{GAP, white in Fig. 41-(B), the additional material stays in place}\} \cup B_a^3 \times [-\varepsilon, +\varepsilon],$$

and correctly means here glued to  $h(S \times 1)$  from the Figure 40. At the same times, the {LHS of the GAP, same Figure 41-(B)} is left in place. In view of the PRODUCT PROPERTY of  $\hat{\Lambda}_1^4$ , as expressed by (132.3), this step, as performed, preserves the global PRODUCT PROPERTY of LAVA.

The Figure 44 should suggest how we relocate CORRECTLY the bottom  $S_0 \times [0, 1]$  of  $\Sigma_1$ . In terms of the pairs described in the formula (132.3), we have a

$$\Sigma_1(\text{REPOSITIONED}) \equiv \Sigma_1(\text{first REDEFINED and then RELOCATED}).$$

(133.II) Next, we **fill the GAP**. The RED band in the Figure 44 corresponds to the  $\Sigma_1(\text{REPOSITIONED})$  above (which might be suggested by the Figure 43 too), which at the time of the Figure 44 in question is just a DRAWING ON TOP OF LAVA  $\hat{\Lambda}_1^4$ . This drawing is, partially, on top of the {additional material, which is neither deleted nor moved}. In the Figure 44, this part of the ride is the area  $[\bar{\alpha}, \bar{\beta}, \gamma, \delta]$ .

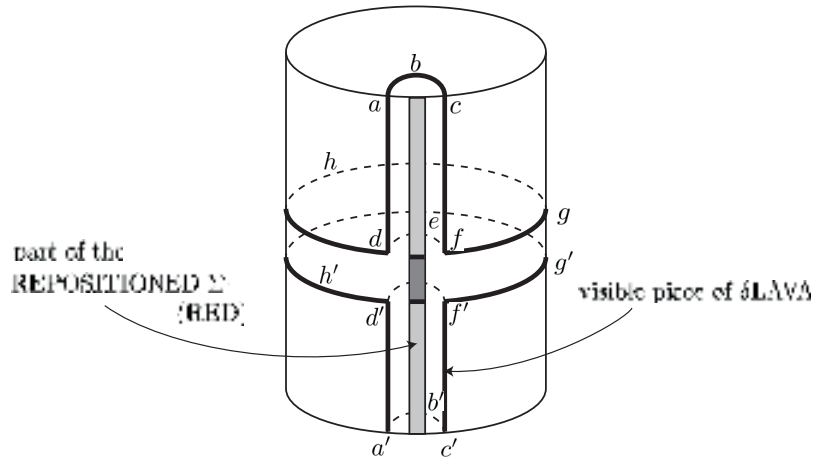


Figure 44.1.

A detail of Figure 44 is being presented here, with one dimension less. We see the red band riding on top of LAVA  $\hat{\Lambda}_1^4$ , precisely over  $\partial\hat{\Lambda}_1^4 - \delta$  LAVA (the **same**  $\delta$  LAVA as in Figure 40, not affected by our DELETIONS and REPOSITIONINGS). Here  $\delta$  LAVA  $\equiv \delta L_0^4$ .

The point here is also that our red drawing (the  $\Sigma_1$  (REPOSITIONED)) does not touch the

$$\delta L_0^4 = \delta(\text{LAVA}(t=0), \text{ with bridges and dilatations included}),$$

which keeps its integrity intact during the DELETIONS and RELOCATIONS, **except** at the singular spot  $S_0 \times 1$ , where it glues to the active 1-handle.

So, with our filling of the GAP, in the present step (133.II) we have a pair

$$([\Lambda_1^4 \cup \{\text{additional material}\}] \underbrace{\cup}_{\Sigma_1 \text{ (REPOSITIONED)}} \text{GAP}, \delta L_0^4)$$

which, in the one hand, manifestly has the PRODUCT property and on the other hand is also isomorphic to the pair which actually interests us  $\{(\text{LAVA}, \delta \text{LAVA})$  at the end of the slide of the  $y = B_a^3 \times [-\varepsilon, \varepsilon]$  which is NON-LAVA over  $x = B_p^3 \times [-\varepsilon, \varepsilon]$ , which is LAVA $\}$ , i.e. LAVA  $(t = \frac{1}{2})$ .

[The formula above is to be compared with a very similar decomposition occurring inside (132.3), but the two  $\Sigma_1$ 's are not quite the same.]

Once all this has been achieved, we can also bring  $\Gamma_j$  to its correct final position, the one suggested in Figure 44.

**Additional explanations.** A) In the Figure 44.1, the fat contours  $[a, b, c, f, g, h, d, a]$  and  $[a', b', c', f', g', h', d', a']$  bound  $\delta L_0^4$ , which, in the context of our figure is 2-dimensional. It splits the presently 3-dimensional LAVA  $\hat{\Lambda}_1^4$ , on top of which the red repositioned  $\Sigma_1$  rides, from the rest. The doubly shaded red zone in Figure 44.1 corresponds to the place where the repositioned  $\Sigma_1$  in Figure 44 goes through  $B_a^3 \times [-\varepsilon, \varepsilon]$ , i.e. through  $\partial \text{LAVA} - \delta \text{LAVA}$ .

B) At the fat site  $[\alpha\beta]$  on the  $\oplus$  side, in Figure 44, we have a contact

$$(\delta \text{LAVA} \mid X_b^2) \cap \text{LAVA} \mid X_r^2$$

where the two things glue together. This is OK (for the product property), since the big collapse  $\text{LAVA}^\wedge \rightarrow \delta \text{LAVA}$  goes (in terms of  $2X_0^2$ ) from the BLUE side to the RED side. (We will come back to this later.) The  $\eta_i$  (green) is the boundary of an exterior disc, so it does not “stick” at  $[\alpha'\beta']$ , where there is no problem. End of explanations.

**Lemma 17. (The time  $t = \frac{1}{2}$  compactification.)** 1) *Proceeding with a finite number of steps like the one described above, starting from the schematical Figure 39, we can realize geometrically the abstract balancing  $(2\Gamma(\infty), \Gamma(1)) \implies (2\Gamma(\infty)[\text{balanced}], \Gamma(3))$  and we will use now the notation  $2\Gamma(\infty)[\text{balanced}] \equiv 2\Gamma(\infty)$  (time  $t = \frac{1}{2}$ ). So inside  $N_1^4(2X_0^2)^\wedge$  we have a transformation proceeding via 1-handle slidings respecting (126.1), (126.2)*

$$(134) \quad (N^4(2\Gamma(\infty)), N^4(\Gamma(1))) \implies (N^4(2\Gamma(\infty)) \left( t = \frac{1}{2} \right), N^4(\Gamma(3))).$$

*This transformation brings us from time  $(t=0)$  to time  $(t = \frac{1}{2})$ , as far as the 1-skeletons are concerned. It also drags along, via covering isotopy, the link (97), hence the transformed family of curves  $\sum_1^\infty C_i$  as well as the  $\sum_1^\infty D^2(C_i)$  which cobounds it. This brings about the LAVA  $(t = \frac{1}{2})$ , which has the STRONG PRODUCT PROPERTY just like LAVA  $(t=0)$ .*

2) *The transformation from (134) leaves the  $\sum_{i=1}^M H_i^b = \sum_1^M b_i$  intact. From now on, this is **the** family of 1-handles of  $N^4(\Gamma(3))$  and we will forget about the  $\sum_1^M H_i^r$ , completely, in the rest of this paper.*

We have a diffeomorphism

$$(135) \quad \left( N^4(\Gamma(3)), \sum_1^M H_i^b \right) \stackrel{\text{DIFF}}{=} \#_{i=1}^M (S_i^1 \times B_i^3, (*) \times B_i^3),$$

i.e. the pair in the LHS of (135) is standard.

3) As a consequence of (122) there is a next diffeomorphism which, until further notice, supersedes the (122)

$$(136) \quad \Delta^4 = N^4(\Delta^2)_{\text{Schoenflies}} \stackrel{\text{DIFF}}{=} N^4(2X_0^2)^\wedge \left( t = \frac{1}{2} \right) = \left[ \left( N^4(2\Gamma(\infty)) \left( t = \frac{1}{2} \right) = \sum_1^\infty h_i \left( t = \frac{1}{2} \right) \right) \cup \right. \\ \left. \cup \text{LAVA} \left( t = \frac{1}{2} \right) \right]^\wedge + \sum_1^{\bar{n}} D^2(\Gamma_j) \subset N_1^4(2X_0^2)^\wedge = \Delta^4 \cup \partial\Delta^4 \times [0, 1].$$

[In order to deduce the (136) from (122), the passage  $(t = 0) \Rightarrow (t = \frac{1}{2})$  has to be conceived in the obvious naïve manner, and NOT like in the context of the Figures 41 to 44.1 (that was necessary just for making sure of the PRODUCT PROPERTY of LAVA  $(t = \frac{1}{2})$ ), and the covering isotopy theorem can then be applied.]

4) We also have a diffeomorphism

$$\left\{ \left[ \underbrace{\left( N^4(2\Gamma(\infty)) \left( t = \frac{1}{2} \right) - \sum_1^\infty h_i \left( t = \frac{1}{2} \right) \right) \cup \text{LAVA} \left( t = \frac{1}{2} \right)^\wedge}_{\text{call this } \hat{Z}^4(t=\frac{1}{2})} \right], \sum_1^M \{ \text{extended cocore } H_i^b \}^\wedge \right\} = \\ \stackrel{\text{DIFF}}{=} \#_{i=1}^M (S_i^1 \times B_i^3, (*) \times B_i^3), \text{ i.e. the pair in the LHS is standard.}$$

So, the  $\sum_1^M \{ \text{extended cocore } H_i^b \}^\wedge$  **is** the system of 1-handles of  $N^4(2X_0^2)^\wedge$  from (136).

5) In the same context of (136) we have the system of exterior discs, which is smoothly embedded

$$(138) \quad \left( \sum_1^M (\delta_i^2, \partial\delta_i^2 = \eta_i(\text{green})) \right) \xrightarrow{J} (\partial\Delta^4, \partial\Delta^4 \times [0, 1]).$$

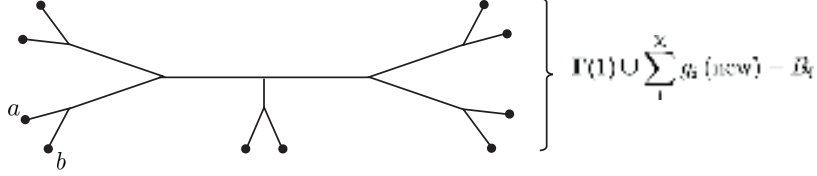
6) (Reminder) We have  $\eta_i(\text{green}) \cdot \left( B - \sum_1^M H_i^b \right) = 0$  but the geometric intersection matrix of interest for us is now the

$$(139) \quad \eta_i(\text{green}) \cdot \{ \text{extended cocore } H_j^b \}^\wedge = \delta_{ij} + \{ \text{parasitical RED terms } \eta_i(\text{green}) \cdot h_k(t = \frac{1}{2}), \text{ where } h_k(t = \frac{1}{2}) \subset \{ \text{exterior cocore } H_j^b \} \}, \text{ here, of course, } 1 \leq i, j \leq M, \text{ and } h_k \in R - B.$$

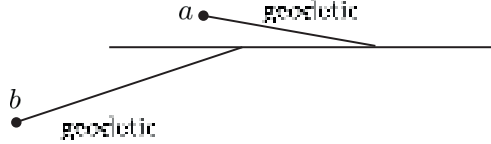
**Sketch of proof.** Enough has been said, since the beginning of the present section VII, so as to make point 1) completely clear. Concerning (135), notice the following, at the purely abstract level of the BALANCING LEMMA 15. Every edge  $e \subset \Gamma(1) = \Gamma(1) \times (\xi_0 = -1)$  carries a  $b_i \in B_0$ , and this is the family  $\sum_1^M b_i$ .

So  $\Gamma(1) - \sum_1^M b_i = \Gamma(1) - B_0 = \sum_{\alpha=1}^{\chi+1} \chi_\alpha$  is a collection of very small graphs  $\chi_\alpha$ , each with a single vertex and each with two ends, corresponding to two distinct  $b_i, b_j$ , with  $1 \leq i, j \leq M$ . When the mechanism which

proves Lemma 15 is applied to this situation, then we get a tree  $\Gamma(1) \cup \sum_1^X g_i(\text{new}) - B_0$ , the vertices of which come organized in pairs, each pair corresponding to the endpoints of one of the BLUE handles in  $\sum_1^M H_i^b$ , like in the drawing below:



Here,  $a, b$  are endpoints of some given  $b_i$  and in real life they do not have to be as close to each other as in the drawing above, but rather like in the next drawing:



The (135) immediately follows from this. Then, as already said, the (136) in 3) follows from  $\{(122) \text{ from } (t=0)\}$ , by appealing to the covering isotopy theorem, and making use of the PRODUCT property of LAVA  $(t=\frac{1}{2})$ .

The basic fact

$$\left( \hat{Z}^4 \left( t = \frac{1}{2} \right) \cup \text{LAVA} \left( t = \frac{1}{2} \right)^\wedge, \sum_1^M \{ \text{extended cocore } H_i^b \}^\wedge \right)_{\text{DIFF}} \overset{\#}{=} \prod_{i=1}^M (S_i^1 \times B_i^3, (*) \times B_i^3)$$

follows by combining (135) with the strong product property of LAVA  $(t=\frac{1}{2})$ . The rest of the Lemma 17 is essentially a reminder, and our proof ends here.  $\square$

The rest of this paper will be devoted to the elimination of the parasitical red terms in (139). We will call this last step

$$(140) \quad \left( t = \frac{1}{2} \right) \xRightarrow{\text{THE COLOUR-CHANGING}} (t = 1).$$

Here are, to begin with, two remarks concerning the parasitical terms  $\{k_k(t=\frac{1}{2}) \text{ from (139)}\}$ .

•) Since the total length of  $\sum_1^M \eta_i(\text{green})$  is finite, i.e.  $\text{length} \left( \sum_1^M \eta_i(\text{green}) \right) < \infty$ , there are, to begin with, only **finitely** many parasitical RED terms in (139).

••) Since  $\eta_i(\text{green}) \cdot \left( B_1 - \sum_1^M H_i^b \right) = \emptyset$ , these terms are all in  $R_0 - B_0 = R_1 - B_1$ .

We go back for a while now to  $t=0$  and we will denote by  $h(r)$  the  $h_i \subset X_0^2 \times r (\approx X_0^2[\text{new}]) \subset 2X_0^2$  and by  $h(b)$  the others, i.e. the  $h_i \subset X_b^2 \subset 2X_0^2$  (see here (16)). This gives a partition

$$(141) \quad \sum_1^\infty h_i = \sum_1^\infty h_\ell(r) + \sum_1^\infty h_k(b)$$

$$\sum_1^\infty C_i = \sum_1^\infty C_\ell(r) + \sum_1^\infty C_k(b).$$

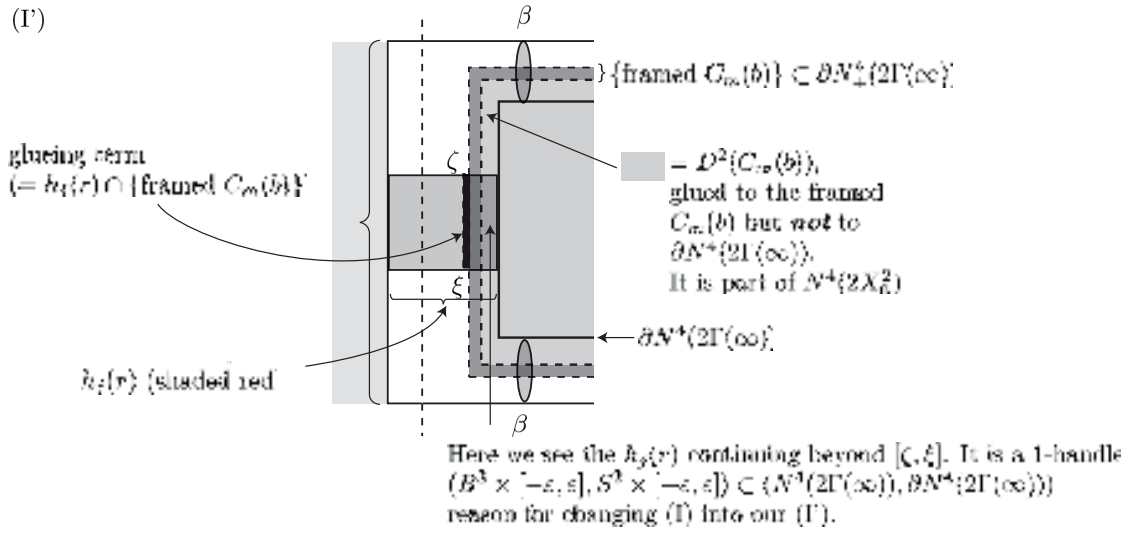
This is Figure 7, embellished. In particular, we see here LAVA ( $r$ ) and LAVA ( $b$ ). Do not mix up notations like  $c(r)$  (going back to the rectangles of Figure 7) and  $C(r)$  occurring in (141). For instance, the  $c(r)$  in (I) is a  $C_k(b)$  because its dual  $h$  is  $h_k(b)$ . The curves dual to the  $h_k(b)$ 's in Figures (II), (III) are  $C_k(b)$ 's living in  $X_b^2$ . The  $h_k(b)$  in (IV) is the dual of  $c_k(r) \in \bigcup_1^\infty C_j(b)$ .



The Figure 7-bis becomes here like (II) or (III), EXCEPT that: (i) There is no longer any LAVA ( $r$ ) contribution and, moreover, (ii) There is no longer any added  $2^d$  meat alive in the purely 1-dimensional  $2\Gamma(\infty)$  | (Figure 7-bis). Very importantly too, at the free LAVA ( $b$ ) site (= face in (II), (III)), the internal collapse LAVA ( $b$ )  $\searrow \delta$  LAVA ( $b$ ) can start. The sign “ $\emptyset$ ” in (II), (III) means, of course, that the corresponding disc is not there in  $2X_0^2$  (but only in  $2X^2$ , where it is present).

LEGEND:  (RED) = LAVA $_r$ ,  = LAVA $_b$ ,  $\zeta$  —  $\xi$  (occurring **only** in (I)) = the GLUEING TERM, explained in the main text.

In the drawing (I') below, a detail of (I) is a bit more accurately rendered.



We will enlarge now the purely 1-dimensional  $2\Gamma(\infty)$  with some  $2^d$  meat, as it is suggested in Figure 45, and all this is eventually part of  $N^4(2\Gamma(\infty))$ . With the forced confinement conditions (101.1-bis), we have set that  $\Sigma c(r) \subset \partial N^4_+(2\Gamma(\infty))$  and Figure 45 should clarify this. We actually have

$$(142) \quad \sum_1^\infty \{\text{little } C_i(\text{or } \Gamma_j)\} + \sum_1^\infty \eta_i \times b + \sum_1^\infty c_k(r) \subset \partial N^4_+(2\Gamma(\infty)),$$

but then,  $\Sigma\{\text{little } C_i\} \subset \sum_1^\infty C_\ell(r)$ , while

$$\sum_1^\infty \eta_i \times b + \sum_1^\infty c_k(r) \subset \sum_1^\infty C_n(b).$$

On the other hand, since the duals of the  $\{\text{little } C\}$ 's are in  $R \cap B$ , they will not contribute much to the present story. Finally, we have

$$\sum_1^\infty C_n(r) = \underbrace{\sum_1^\infty \{C_i(\text{remaining})\}}_{\text{in } \partial N^4_-(2\Gamma(\infty))} + \underbrace{\sum_1^\infty \{\text{little } C_i\}}_{\substack{\text{in } \partial N^4_+(2\Gamma(\infty)) \\ \text{(by our construction)}}}.$$

We will denote by  $\{\text{little } h_i\} \subset \sum_1^\infty h_\ell(r)$  the dual of  $\{\text{little } C_i\}$ . Since both  $\Gamma(\infty) - R$  and  $\Gamma(\infty) - B$  have to be trees, this forces the  $\{\text{little } h_i\}$  to be in the family  $R_0 \cap B_0$ . So, in Figure 45-(II) it could (sometimes) happen that the RED  $h_j(r)$  we see is a  $\{\text{little } h_i\}$ . The dual curve  $\{\text{little } C_i\}$  occurs then as two lines of red colour both inside the  $\oplus$  part, in the two contiguous Figures (II) and (III), so that the two lines should melt in to a single closed curve. See here the (II), (III) in our Figure 45, with the points  $u$  and  $v$  common to both. In terms of Figure 32, the (II) corresponds to the little 1-handle and (III) to the main body. The observant reader will have already figured out that we need two adjacent Figures (III) and three red lines in all, but that does not change our story. But before saying more concerning the pair  $(\{\text{little } C_i\}, \{\text{little } h_i\})$ , I will introduce the pairs

$$(143) \quad (\text{LAVA}_r(t=0), \delta \text{LAVA}_r(t=0)) = \left( \sum_{\ell=1}^\infty (h_\ell(r) \cup C_\ell(r)), \right.$$

$$\left. \delta \text{LAVA}_r(t=0) \equiv (\delta \text{LAVA}(t=0)) \cap \partial \text{LAVA}_r(t=0) \right) = \partial \text{LAVA}_r(t=0) \cap \partial \left[ N^4(2\Gamma(\infty)) - \sum_1^\infty h_\ell(r) \right],$$

and

$$\begin{aligned} (\text{LAVA}_b(t=0), \delta \text{LAVA}_b(t=0)) &= \left( \sum_{k=1}^\infty (h_k(b) \cup C_k(b)), \delta \text{LAVA}_b(t=0) \equiv \right. \\ &\quad \left. \equiv (\partial \text{LAVA}_b(t=0)) \cap \partial \left[ N^4(2\Gamma(\infty)) - \sum_1^\infty h_\ell(b) \right] \right). \end{aligned}$$

Very importantly, there is also a glueing term between the two lavas, namely

$$(144) \quad \text{GLUEING TERM} \equiv \delta \text{LAVA}_b(t=0) \cap \partial \text{LAVA}_r(t=0),$$

illustrated in the Figure 45-(I), when it occurs as  $[\zeta, \xi]$ .

**Lemma 18.** 1) *Individually, each of the two lavas,  $(\text{LAVA}_r(t=0), \delta \text{LAVA}_r(t=0))$  and  $(\text{LAVA}_b(t=0), \delta \text{LAVA}_b(t=0))$  have the STRONG PRODUCT PROPERTY. That property, for the full  $(\text{LAVA}(t=0), \delta \text{LAVA}(t=0))$  is a succession of two infinite collapses  $\text{LAVA}_b(t=0) \longrightarrow \delta \text{LAVA}_b(t=0)$  and  $\text{LAVA}_r(t=0) \longrightarrow \delta \text{LAVA}_r(t=0)$  starting from the free faces of  $\partial \text{LAVA}_r(t=0)$ , which include now the GLUEING TERM.*

2) *We have*

$$\text{LAVA}_b(t=0) \subset \{\text{the } \oplus \text{ side of the SPLITTING of } N^4(2X_0^2) \text{ by } \Sigma_\infty^2\}.$$

The bulk of  $\text{LAVA}_r(t=0)$  lives on the  $\ominus$  side of the SPLITTING by  $\Sigma_\infty^2$ . But then, for obvious geometrical reasons, each  $h_k(r) = B_k \times [-\varepsilon, \varepsilon]$  which is definitely  $\text{LAVA}_r$  **has** to have its  $\ominus$  half and its  $\oplus$  half. For instance, in Figure 45-(I) it is through its  $\oplus$  half that the corresponding  $h_j(r)$  gets on top of its GLUEING TERM. See also Figure 49 when each  $B^3(r(j))$ ,  $B^3(b(j))$  has a  $\ominus$  half and a  $\oplus$  half. This has nothing to do with the 1-handle in question being  $\text{LAVA}_r$  or  $\text{LAVA}_b$ .

Concerning the  $(\{\text{little } C_i\}, \{\text{little } h_i\})$  we have the following

**Lemma 19.** 1) *Since  $\{\text{little } h_i\} \in R_0 \cap B_0$ , it is not concerned by the active part of the COLOUR CHANGE (Lemma 14 and its geometric realization, in the rest of this section).*

2) We have  $(\{\text{little } h_i\} \cup D^2(\{\text{little } C_i\})) \subset \text{LAVA}_r(t=0)$ , and also  $\sum_i (\{\text{little } h_i\} \cup D^2(\{\text{little } C_i\})) \cap \text{LAVA}_b(t=0) = \emptyset$  (no corresponding glueing terms exist).

3) The  $(\{\text{little } h_i\} \cup \{\text{little } C_i\})$  is an endpoint for the RED collapsing flow of  $\text{LAVA}_r(t=0)$ , in the sense that there are no outgoing flow-lines from  $i$ .

All these things which we have just said have concerned the  $\text{LAVA}(t=0)$  and, of course, together with the process  $(t=0) \implies (t=\frac{1}{2})$  which we have described, comes also a transformation  $\text{LAVA}(t=0) \implies \text{LAVA}(t=\frac{1}{2})$  which involves items (141) to (144), in a manner which should be more or less automatic. For instance, in the context of the Figure 40, in the sliding of  $y$  over  $x$ , all the curves which we see on the side of  $\partial N_+^4(2\Gamma(\infty))$  are  $C(b)$ 's, meaning  $\eta \times b$ 's, or  $c(r)$ 's, or of course  $\eta_\ell(\text{green})$ 's OR pieces of  $\{\text{little } C\}$ , in the worst case. Then  $h_i$  is actually RED (precisely an  $R \cap B$ ). So these  $\{\text{little } C\}$ 's are certainly mute in the context of Figures 40 to 49, when they have not been explicitly drawn. They may be assumed far from the moving  $B_a^3 = y$ .

Also, the  $[\alpha, \beta]$ 's in Figure 44 are all new GLUEING TERMS, and there is a large margin of freedom concerning their positioning. We move now to the next geometric step

$$\left(t = \frac{1}{2}\right) \xrightarrow{\text{THE BIG COLOUR-CHANGE}} (t = 1),$$

the aim of which is to realize the BIG BLUE DIAGONALITY condition

$$\eta_i(\text{green}) \cdot \{\text{extended cocore } H_j^b\}^\wedge = \delta_{ij} \text{ AND } \eta_i(\text{green}) \cdot \left(B - \sum_1^M H_j^b\right) = 0.$$

What this means is that we both want to get rid of those finitely many RED parasitical terms  $h_k = h_k(t=\frac{1}{2}) \in R - B$  which occur in (139) and, at the same time, **not** create any additional  $\eta(\text{green}) \cdot B$ . For this purpose, we start by constructing a finite, sufficiently high truncation  $\Phi_1$  of the isomorphism  $\Phi$  (103), a truncation which is such that  $\Phi_1$  is gotten by stopping the inductive process in Lemma 14 at some finite level which we denote by  $j_1$ . This  $j_1$  is high enough so that we should have

(144.1) The  $j_1$  is sufficiently close to  $\infty^{\text{tv}}$  so that  $R(j_1)$  should contain all the  $\Phi(h_k, \text{parasitical term in (139)})$ . Put differently, we want to have  $\{\text{All the } h_k \text{ which are } R - B \text{'s touched by } \sum_1^M \eta_\ell(\text{green}), \text{ off-diagonally}\} \subset \{\text{The family } r(1), r(2), \dots, r(j_1)\} \subset R - B - R(j_1)$ . These  $h_k$ 's are all, automatically  $\text{LAVA}(t)$ , for all  $t$ 's until they change their colour, and then we have

$$\Phi(r(i)) \in B - R, \text{ for } i \leq j_1.$$

So, **all** the parasitical  $h_k$ 's will have been already turned BLUE, at time  $j_1 = (t=1)$ . Their place inside the LAVA will have been taken by the BLUE  $\Phi(h_k(\text{parasitical}))$ , which are **untouched** by  $\sum_\ell \eta_\ell(\text{green})$ . Here are some additional explanations concerning the family

$$\text{"LHS of (144.1)} \equiv \{\text{the parasitical } h_k \in R - B, \text{ touched by } \sum_\ell \eta_\ell(\text{green})\}.$$

- ) Since length  $\sum_\ell \eta_\ell(\text{green}) < \infty$ ,  $\#$  ("LHS of (144.1)")  $< \infty$  too.
- ) Our LHS of (144.1) corresponds to the (139).
- ) In what follows next, in this paper, the family  $\{\text{LHS of (144.1)}\}$  will only **decrease** when  $t$  increases.

(144.2) At any time  $t$ , here is the list of constituent pieces of  $\text{LAVA}(t)$ : 1-handles  $h_i$  or  $\Phi(h_i)$ , RED or BLUE, which are LAVA and the corresponding  $D^2(C_i)$  (dragged over  $\Phi(h_i)$  when  $h_i \Rightarrow \Phi(h_i)$ , and this is part of  $\text{LAVA}_r$ ), then also LAVA BRIDGES and LAVA DILATATIONS.

The total contribution of these additional items will be compact. The additional pieces insert themselves, when that is needed, in order to preserve the PRODUCT PROPERTY OF LAVA, when  $C \cdot h = \text{id} + \text{nil}$  is contradicted, between a  $D^2(C_i)$  and a  $h_j$  where we have  $D^2(C_i) \cdot h_j \neq 0$  (i.e.  $i \rightarrow j$ ) in the geometric intersection matrix.

Any  $h \in R - B$  might be replaced at some time  $t_0(h)$ , and then for all times  $t \geq t_0(h)$ , by the corresponding  $\Phi_1(h) \in B - R$ , after which the  $\Phi_1(h)$  replaced  $h$  inside LAVA, while  $h$  leaves the scene for ever.

This will be the GEOMETRIC 4<sup>d</sup> REALIZATION OF THE COLOUR-CHANGE. End of (144.2)

In the meanwhile, during the step  $(t = \frac{1}{2}) \implies (t = 1)$ , the condition  $C \cdot h = \text{id} + \text{nilpotent}$  will get violated and when we will have to get the {extended cocores}'s, in particular the all-important {extended cocore  $b_{i \leq M}$ } 's, we will have to appeal directly to the PRODUCT PROPERTY of LAVA.

With this, both the contribution of the population of  $h$ 's and of the LAVA bridges inside the extended cocores, may increase in time; but see here also what is said concerning the parasitical  $h_\ell$ 's (see the (139) for these parasites).

The LAVA bridges will be living inside  $\partial N_-^4(2\Gamma(\infty))$  (more precisely their piece of  $\delta(\text{LAVA bridges}) \subset \partial N_-^4(2\Gamma(\infty))$ , far from  $\sum_1^M \eta_\ell(\text{green}) \subset \partial N_+^4(2\Gamma(\infty))$ .

On the other hand, the total population of these  $h_\ell$  **touch**ed by  $\sum_1^M \eta_\ell(\text{green})$  (and they will all be certainly LAVA), will constantly **decrease** in time, during the ministeps of the  $(t = j - 1) \Rightarrow (t = j)$  occurring in the (144.3) below. They will eventually all **turn BLUE**, via the COLOUR-CHANGING process. So, at time  $t = 1$ , as a consequence of the little blue diagonalization already achieved at (95) above, for any  $h_i$  still alive inside LAVA, we will find

$$\sum_{\ell=1}^M \eta_\ell(\text{green}) \cdot h_i = \emptyset.$$

Now, in order to achieve the (144.1), we will realize, for all the  $j \leq j_1$ , geometrically, in 4<sup>d</sup> inside  $N_1^4(2X_0^2)^\wedge$ , the abstract steps  $R(j - 1) \Rightarrow R(j)$  from Lemma 14. This means that we will have a whole collection of intermediary times, between  $t = \frac{1}{2}$  and the final  $t = 1$

$$(144.3) \quad \left(t = \frac{1}{2}\right) \text{ (which we also call } (t = (j = 0)), t = (j = 1), t = (j = 2), \dots, (t = j - 1),$$

$$(t = j), \dots, (t = j_1) \equiv (t = 1).$$

At all the times  $t$  (144.3) the  $N^4(2\Gamma(\infty))$  will be unchanged, with us, but then there will be fluid times  $j - 1 < t < j$ , when  $N^4(2\Gamma(\infty))$ , at the intermediary moments between the ones in (144.3), will change. It will become then a fluid, time-dependent object denoted  $Z(t)$ . But, I insist, at all the integral moments  $t$  in (144.3), we always have  $Z(t) = N^4(2\Gamma(\infty))$ .

We will describe now the main inductive step, namely

**The geometry of the step  $(t = j - 1) \implies (t = j)$ ,  $j \leq j_1$  in (144.3) .**

We insist, like for  $t = 0 \Rightarrow t = \frac{1}{2}$ , that (126.1) and (126.2) should be satisfied. The  $\Gamma_1(\infty) \subset \Gamma(2\infty)$  (to be changed into  $N^4(\Gamma_1(\infty)) \subset N^4(2\Gamma(\infty))$ ), at the initial stage  $t = j - 1$ , is shown in the Figure 46, which is a vastly more elaborate form of the Figure 37. But, before we start looking into the figure, here comes the

**Important comment (144.4) (concerning (126.1)).** We have the following general formula, valid at all times

$$\text{LAVA} = \text{LAVA}_r \underbrace{\cup}_{\text{GLUEING TERMS}} \text{LAVA}_b.$$

Independently of each other, both  $\text{LAVA}_r$  and  $\text{LAVA}_b$  share the strong product property. With this, however we place our GLUEING TERMS, the **global** lava has automatically the STRONG PRODUCT PROPERTY, with a collapsing flow starting in  $\text{LAVA}_b$ , reaching  $\text{LAVA}_r$  at the GLUEING TERMS, now free faces, and continuing through the  $\text{LAVA}_r$ . When we want to check (126.1) at various stages of our construction, we should keep this in mind.

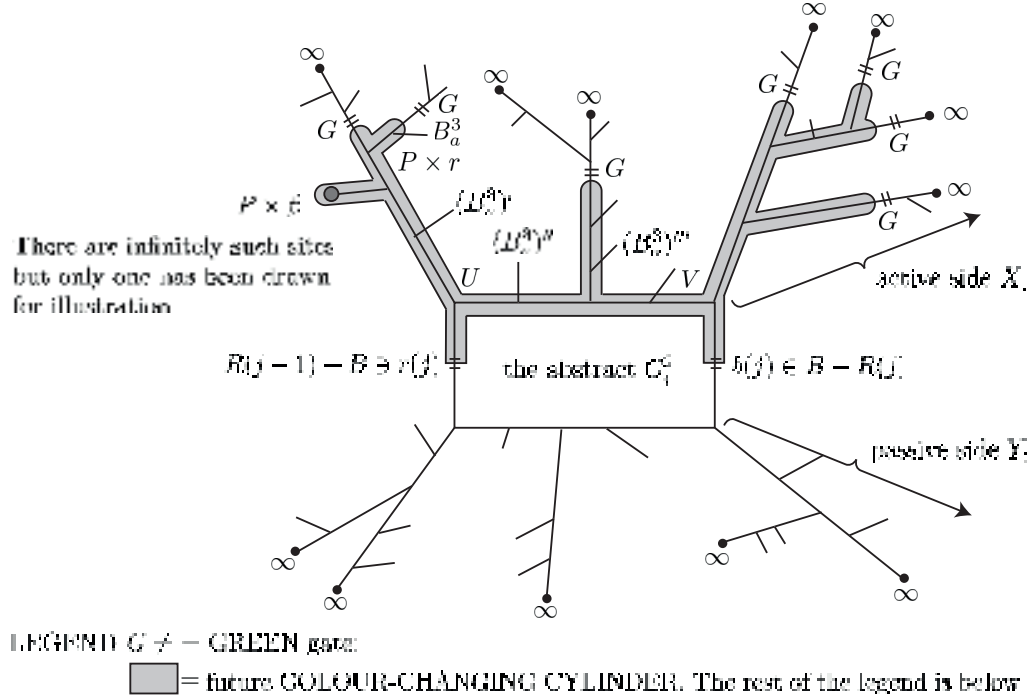


Figure 46.

We see here the  $\Gamma_1(\infty) (t = \frac{1}{2}) = \Gamma_1(\infty)[\text{balanced}] \subset 2\Gamma(\infty) (t = \frac{1}{2})$ , which we will just denote by  $\Gamma$  for typographical simplicity. At stage  $j-1$  of the inductive process in the ABSTRACT COLOUR-CHANGING Lemma 14, realized now geometrically in  $4^d$ , this is what we will find, just before  $j-1 \Rightarrow j$  starts. So, what we have displayed, in the drawing above, is

$$\{\text{the tree } \Gamma - R(j-1) = X_j \cup b(j) \cup Y_j\} \cup \{\text{the edge } r(j)\}.$$

Remember the  $R \cap B$  is mute, in particular so are the  $\{\text{little } h_i\}$ 's.

So we need not worry about  $R \cap B$  in the process

$$(*) \quad \left(t = \frac{1}{2}\right) \implies (t = 1)$$

which implements geometrically the COLOUR CHANGE, leading to Lemma 20.

The  $\Gamma(3)$ , result of the BALANCING, is by now already with us and the  $\Gamma(3) - R(j-1)$  lives deep inside the passive part  $Y_j$ , which is untouched by the action in  $(*)$ . It is exactly this presence of  $\Gamma(3)$  inside one (and only one) of the  $X$  or  $Y$ , which decides which is the active  $X_j$  and which is the passive  $Y_j$ . The LEGEND continues now:  $\text{---}$  (= short RED arc) = loose end of red arcs in  $R(j-1) - \{r(j)\}$ . Each such arc comes with two loose ends and by joining all these pairs together, we can reconstruct our  $\Gamma(\equiv \Gamma_1(\infty)[\text{balanced}]$  at stage  $j-1$ );  $\#$  = green GATE, to be explained in the main text. One has also drawn as a sample a  $P \times [r, \beta]$ , but there are many more such; they open the road going from our  $\Gamma_1(\infty)$  to the whole of  $\Gamma(2\infty)$ .

Here are some additional EXPLANATIONS for Figure 46. We have finitely many sites denoted  $G = \text{“GREEN GATE”}$ ; these are not in  $R \cup B$ , reason for a different COLOUR, but they should be compared with the RED and/or BLUE 1-handles.

Here is what the finite system of GATES  $\{G\} \subset X_j$  does for us.

(145.1) It separates a compact part of  $X_j$  from  $\infty^{\text{ty}}$ . This compact part is the one touching to  $b(j), r(j)$  and the intermediary connecting part between them. Inside it, we will create the COLOUR-CHANGING CYLINDER.

(145.2) The system  $\{G\}$  is sufficiently close to  $\infty^{\text{ty}}$  so that we should have  $\{\text{extended cocore } G\} \cap r(j) = \emptyset$ .

One should use here the following general property which, for illustration, I state in terms of  $N^4(2\Gamma(\infty)) \subset N_1^4(2X_0^2)^\wedge$ . Let  $S_n \subset 2\Gamma(\infty)$  be a site which is such that  $S_n$  possesses an  $\{\text{extended cocore } S_n\} \subset N^4(2X_0^2)$  and such that  $\lim_{n=\infty} S_n = \infty$ , in  $\Gamma(2\infty)$ . Then, we also find that

$$\lim_{n=\infty} \{\text{extended cocore } S_n\} = \infty, \text{ in } N^4(2X_0^2).$$

Moreover, if one adds here all the appropriate quantifiers, there is then also a UNIFORMITY property too, and this yields what we want above.

Let us be even a bit more precise. In the same vein as above, we can ask that, whenever there is a contact  $\{G\} \cap C_\ell \neq \emptyset$  with, let us say,  $C_\ell = C_\ell(r)$ , then in the RED order one has  $h_\ell > r(j)$ . Next, we have

(145.3) With  $b(j)$  and  $r(j)$  being given, the distinction between  $Y_j$  being passive and  $X_j$  being active depends on the  $\Gamma(3) - R(j-1)$  being inside  $Y_j$ , which fixes the distinction. Then  $\{G\}$  is located in  $X_j$ . With  $r(j) + b(j)$  deleted,  $X_j$  and  $Y_j$  are no longer connected with each other.

(145.4) We can choose our family  $\{G\}$  sufficiently close to  $\infty^{\text{ty}}$  so that, we not only should have the (145.2) above, but also the following feature

$$\{G\} \cap \Sigma_{\eta_\ell(\text{green})} = \emptyset.$$

We use here the fact that  $\Sigma_{\eta_\ell(\text{green})}$  is finite. Also, each  $G$  is a 1-handle not in  $R \cup B$ , of its new colour “GREEN”, so that, like for the  $B_a^3$  in the Figure 40 we have

$$B^3(G) = \frac{1}{2} B^3(G)(+) \underbrace{\cup}_{\sigma^2} \frac{1}{2} B^3(G)(-)$$

and, a priori, we could have  $\partial N_+^4(2\Gamma(\infty)) \supset \frac{1}{2} N^3(G)(+) \cap \eta_\ell(\text{green}) \neq \emptyset$ , something which we want to avoid, and **will avoid**, by chosing  $\{G\}$  very close to  $\infty^{\text{ty}}$

(145.4-bis) The  $\{G\}$  induces a finite partition

$$X_j = X_j^0 \cup_G X_j^1 \cup \dots \cup_G X_j^\rho$$

where each  $X_j^\varepsilon$  is a tree,  $X_j^0$  being finite and the others infinite.

This family  $\{G\}$  serves to allow for a **finitistic** step  $j-1 \Rightarrow j$ ; without it, the step in question would require an infinite process, all by itself. Now, in order to perform our inductive step  $j-1 \Rightarrow j$ , on the road from  $t = \frac{1}{2}$  to  $t = 1$ , we will have to break it into three successive parts, namely the STEPS I, II and III, to be described below.

**Important comments (145.4-ter).** At the level of Figure 46 all the 1-handles  $R \cap B$  have been cut, like in the 1) of the ABSTRACT COLOUR-CHANGING LEMMA 14. This means that there is no contribution

of {Figure 45-(II) with {little  $C$ } present}, to our figure. The {Figure 45-(III) with {little  $C$ } (but without the  $h_j(r) \in \{\text{little } h\} \in B \cap R\}$  is present, and we see its reflex in the upper side of the Figure 49. But, in Figure 46, the  $R \cap B$ 's, including our {little  $h$ } 's are mute.

**The Step I.** In terms of the Figure 46, but now geometrically in  $4^d$ , this consists in the following slidings of 1-handles over other 1-handles, in succession:

a) We start with a first succession of elementary steps, for which we consider the various

$$B_a^3(\text{LAVA}) \in \{\text{the short red arcs sticking out of } X_j^0 \text{ (145.4-bis)}\}.$$

These  $B_a^3$ 's are a finite family of 1-handles presenting themselves like th  $B_a^3$  in the Figure 39-(II), except that contrary to that figure, these  $B_a^3$ 's are now LAVA.

The  $B_a^3$ 's have to slide over  $B_p^3 = r(j)$  which is LAVA too.

b) Next, comes a second succession of elementary steps, where each individual  $G = B_a^3$  (non LAVA) slides over the same  $B_p^3 = r(j)$  (LAVA), as above.

Since the case b) is somewhat simpler than the case a), we will describe it first.

**The slide of  $G$  (non-LAVA) over  $r(j-1)$  (LAVA).** This is very much like in the Figures 40 to 44, which can be used again, with the obvious change  $y \Rightarrow G = \text{our new } B_a^3$ ,  $x \Rightarrow r(j) = \text{our new } B_p^3$ . Condition (124) which implied that  $\{\text{extended cocore } y\} \cap x = \emptyset$  is replaced now by the (145.2). Otherwise, with our obvious changes of notation, this step is to be described by the same Figures 40 to 44 as before, and we leave it at that, as far as the detailed description of the step is concerned, except for the following COMMENT (Caveat!). The (126.4) is now no longer with us, so that in the Figure 40 for  $G$ , which we have not drawn, there could happily be BLUE spectators, like in the schematical Figure 39. But, because of (145.4) this **cannot** come with unwanted contacts  $\eta_\ell(\text{green}) \cdot \left(B - \sum_1^P H_i^b\right)$  which would contradict our little BLUE DIAGONALIZATION. [The  $G$ 's can be chosen sufficiently close to  $\infty^{\text{ty}}$  so that  $(G \cup (X_j^1 \cup \dots \cup X_j^p)) \cap \Sigma \eta_\ell(\text{green}) = \emptyset$ .]

We will describe now the sliding of the finitely many  $B_a^3(\text{LAVA}) \equiv \{\text{any of the small RED arcs shooting out of } X_j^0 \supset [U, V], \text{ in the Figure 46}\}$ . Keep in mind that this slide of  $B_a^3$  (LAVA) over  $B_p^3(\text{LAVA}) = r(j)$  is to take place before the sliding of the Gates  $G$  (non LAVA), over the same  $B_p^3$  (LAVA). It is only for expository purposes that the slide of the  $G$ 's was discussed first.

**The slide of  $B_a^3$  (LAVA) over  $B_p^3$  (LAVA)  $= r(j)$ .** At this point there is a big complication occurring in our story. Both  $B_a^3$  and  $B_p^3$  are members of the family  $\{h\} \cap (R - B)$  and in the RED order, both cases

$$B_p^3 > B_a^3 \quad \text{OR} \quad B_a^3 > B_p^3$$

are possible. Now, when  $B_p^3 > B_a^3$ , then the sliding of  $B_a^3$  over  $B_p^3$  brings about a VIOLATION of our so far sacro-sancted feature  $C \cdot h = \text{id} + \text{nilpotent}$ , which we will have to drop now. Actually, from now on, we will have to live without it.

The condition  $C \cdot h = \text{easy id} + \text{nilpotent}$ , is actually less sacro-sancted than confinement, a condition which will never get violated. We still want to perform the present step so that the conditions (126.1) and (126.2) should be respected.

The point here is that, as long as lava continues to have its STRONG PRODUCT COMPANY, we can define extended cocores using that, without invoking  $C \cdot h = \text{id} + \text{nilpotent}$ .

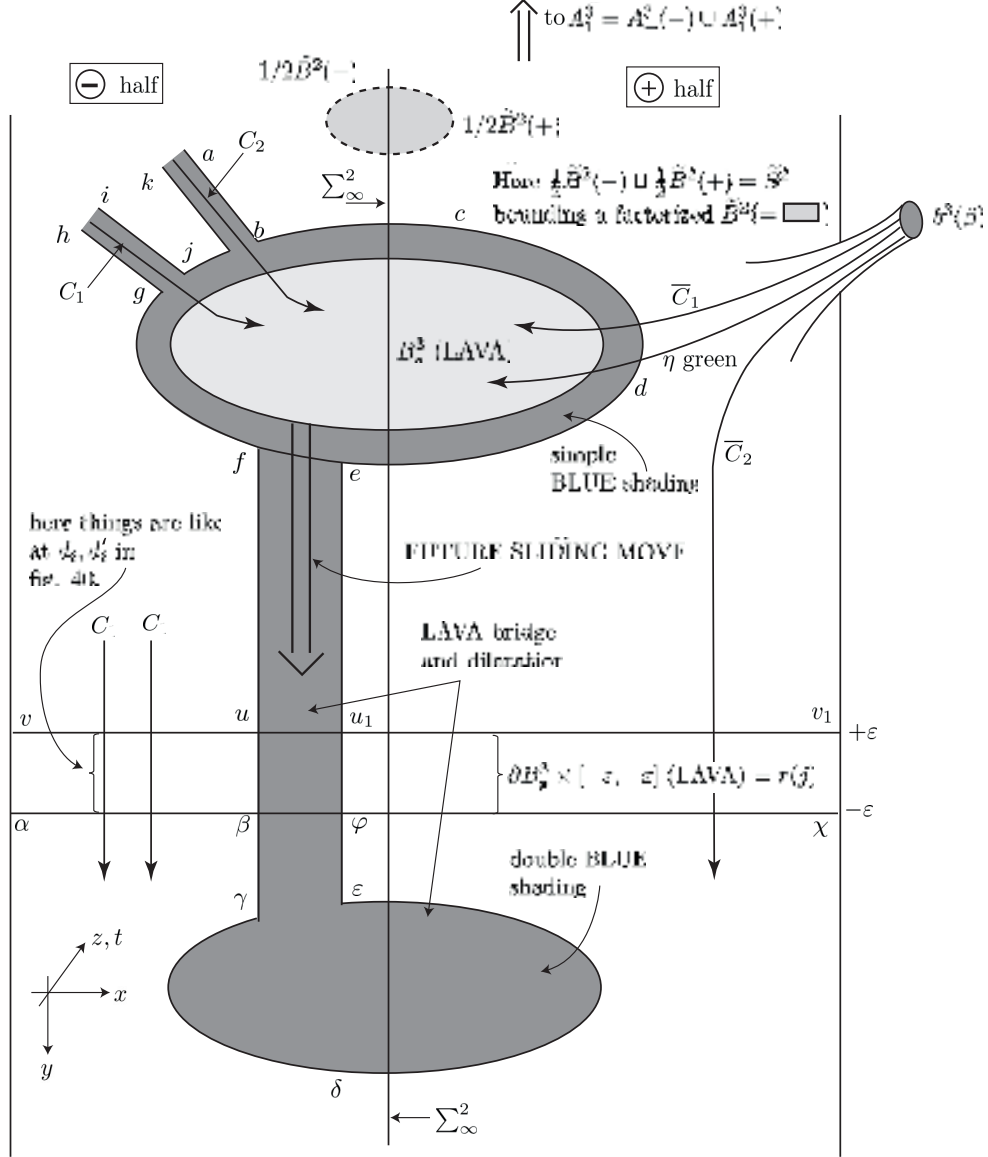


Figure 47.



We see here the beginning of the slide of  $B^3_a(\text{LAVA})$  over the  $r(j) = B^3_p(\text{LAVA})$ . For the sake of typographical simplicity we do not use here the realistic dimensions, like in Figure 40, with which this figure should be compared, but we proceed more schematically, with one dimension less. This figure is supposed to stand for a relevant part of  $\partial N^4(2\Gamma(\infty))$ . More will be said concerning it in the main text. The  $B^3_a(\text{LAVA}) \cup C_1 \cup C_2$  are drawn in RED.

For our present sliding of  $B^3_a(\text{LAVA})$  we will proceed like in the Figure 40, but paying special attention to LAVA, which includes now  $B^3_a$ . Figure 40 is replaced by the Figure 47. And the only things in this figure which are supposed to be alive, before our action starts, are the RED  $B^3_a(\text{LAVA})$ , the passive  $B^3_p(\text{LAVA}) \times [-\varepsilon, \varepsilon]$




and the various curves riding over them (only the ones riding over  $B_a^3$  are actually drawn). Except for the  $\eta(\text{green})$ 's all the other curves living on the  $\ominus$  side (respectively on the  $\oplus$  side) are part of  $\text{LAVA}(r)$  ( $t = \frac{1}{2}$ ), respectively of  $\text{LAVA}(b)$  ( $t = \frac{1}{2}$ ).

Then a preliminary action starts (and we are here still at the level of Figure 47):

a) The original red LAVA extends to the shaded blue area  $[a, b, c, d, e, f, g, h, i, j, k]$  with the RED  $B_a^3$  sitting now on top of it, dragging along the various curves like  $C_1, C_2, \bar{C}_1, \eta(\text{green})$  which climb on  $B_a^3 \times [-\varepsilon, +\varepsilon]$ . At this point our contour  $[a, b, \dots, k]$  above is in  $\partial \delta \text{LAVA}$ , with  $\delta \text{LAVA}$  under the whole shaded area (with the shaded red (  ) *sitting on top of* the shaded blue (  )).

[Here is the scenario for the upper part of Figure 47. In the beginning there is only the RED part. Then the blue  $[j, b, a, d, e, g, j]$  is glued to it; and the red blob climbs on it, sliding. Then, finally along the curve  $C_1, C_2$  and UNDER THEM, we add the blue thin shaded zones all along their free parts. Importantly, all that red part is now disconnected from  $\delta \text{LAVA}$ .]

b) Next, like in Figure 40 a LAVA BRIDGE and LAVA dilatation grows out of the a) above. This is doubly shaded (  ) and when this doubly shaded area meets  $[\partial B_p^3 \times [-\varepsilon, \varepsilon](\text{LAVA})]$ , it melts into it, like in Figure 40. To be very explicit, by now  $\partial(\delta \text{LAVA})$  has become the disjointed union

$$\{[a, b, c, d, e, u_1, v_1] \cup [k, j, i] \cup [h, g, f, u, v]\} + \{[\alpha, \beta, \gamma, \delta, \varepsilon, \varphi, \chi]\}.$$

Notice that there are no curves  $\Gamma_j$  in the Figure 47. Normally they are in  $Y_j$ , out of the universe of our figure. But one way or another, this is not something to bother us. A piece of curve  $\Gamma_j$  surviving inside the  $X_j$  of Figure 46 and climbing on the  $B_a^3$  (Figure 47), can be treated now on par with the  $C_i$ 's; there is now no GAP like in Figure 41-(B), which had forced a different treatment of the  $\Gamma_j$ 's with respect to the  $C_i$ 's, before.

Now the real action of sliding  $B_a^3(\text{LAVA})$  over  $B_p^3(\text{LAVA}) = r(j)$ , can take place, and Figure 48 refers to this slide.

The important point is that, from the viewpoint of  $(\text{LAVA}, \delta \text{LAVA})$ , the sliding of the  $B_a^3(\text{LAVA})$  over  $B_p^3(\text{LAVA})$  is now an *isotopic internal* to LAVA. So, although we have lost the feature  $C \cdot h = \text{id} + \text{nilpotent}$ , the LAVA retains its STRONG PRODUCT PROPERTY. Moreover, the (144.2) stays with us too.

Very importantly, when moving from Figure 47 to Figure 48, the  $\delta \text{LAVA}$  stays unchanged. Very importantly too, the  $\{\text{extended cocores } S\}$  (for "site"  $S$ ), in particular the all-important  $\{\text{extended cocore } H_{j \leq M}^b\}$ , are *defined* from now on, not by appealing to  $C \cdot h = \text{id} + \text{nil}$  (which is, in principle, no longer with us), but *directly to the* PRODUCT PROPERTY OF LAVA.

Here is an additional explanation, concerning our Figures 47, 48.

In the context of Figure 47 we may happily have BLUE spectators  $B$ , and contrary to that had happened for  $G$ , this may come (at the level of Figure 48) with *unwanted contacts*

$$(145.5) \quad (\text{BLUE spectator } B) \cap \eta_\ell(\text{green}) \neq \emptyset,$$

which are a temporary violation of the little BLUE DIAGONALIZATION, and to which we will have to come back later on.

It is the STEP III which will take care of this potentially dangerous complication. In the same vein, our move may also introduce new contacts, very much unwanted too (see Figure 48-(A))

$$(145.6) \quad (B_p^3 = r(j)) \cap \eta_\ell(\text{green}) \neq \emptyset,$$

exactly as consequence of the  $B_a^3 \cap \eta_\ell(\text{green})$  from Figure 47.



All this will be taken care of below. [The BLUE spectators mentioned above, and which are like the  $b$ 's in the Figure 39, come exactly from  $B \cap X_j^0 - \{b_j\}$ , see here Figure 46 and formula from (145.4-bis).]

**Step II.** As a result of STEP I, our  $N^4(2\Gamma(\infty))$  has been changed into a fluid object, the time-depending  $Z^4(t)$ , where  $j-1 \leq t \leq j$ . The net result of our STEP I (first part of  $j-1 \Rightarrow j$ ), is that at the time

$$\{t = j - 2\varepsilon\} \equiv \{\text{the time } t \text{ when the STEP I above has been completed}\},$$

the situation is the following. Once the  $B_a^3$ 's, i.e. the small RED arcs which stick out of  $X_j^0$  in the Figure 46, and the green gates  $G$  too, have slid over  $r(j)$ , one can put together a COLOUR-CHANGING CYLINDER

$$(145.7) \quad B^3 \times [r, b] \subset Z^4(t = j - 2\varepsilon),$$

connecting  $B^3(r(j))$  to  $B^3(b(j))$  and which is factorized too, i.e. divided naturally into  $(\pm)$ -halves. Figure 49 presents the lateral surface of this cylinder

$$(146) \quad S^2 \times [r, b] = \left( \frac{1}{2} B^2(+) \times [r, b] \right) \bigcup_{\Sigma_\infty^2} \left( \frac{1}{2} B^2(-) \times [r, b] \right).$$

In the Figure 46, at its schematical 1<sup>d</sup> level, our (145.7) corresponds to that piece of the graph, resting on  $[r(j), U, V, b(j)]$  and stretching all the way to the gates  $G$ . To make things completely clear, this piece, producing the COULOUR-CHANGING cylinder is surrounded by the shaded area with its boundary  $(\text{=====})$ . After the sliding of the  $B_a^3$ 's and the  $G$ 's, this is separated now from the rest of what has become out of the graph  $\Gamma$  in Figure 46 at time  $t = j - 2\varepsilon$ , by exactly  $r(j)$  and  $b(j)$ . The red arcs which in Figure 46 stick out of the shaded area, are the  $B_a^3(\text{LAVA})$ 's which have already slid over  $r(j)$ .

Our COLOUR-CHANGING CYLINDER is made exactly out of the following spare parts, which in a 1<sup>d</sup> version are vizualizable in Figure 46.

#### List of spare parts of the colour-changing cylinder:

i) The edge  $[r(j), U]$ ; ii) The edge  $[U, V] \cup \{\text{all the part of } X_j \text{ between the } [U, V] \text{ and the GATES } G \text{ (see Figure 46)}\}$ ; iii) The edge  $[V, b(j)]$ . What we have just unrolled corresponds to the  $X_j^0$  in (145.4-bis) and here both the  $B_a^3(\text{LAVA})$ 's and the GATES  $G$  have already slid away, leaving behind them ghostly terms  $\# B^3(G)$ ,  $\# B_a^3$ , suggested in the Figure 49. There are very much like the  $A_1^3(-) \cup_{D^2} A_1^3(+)$  in Figure 40.

With this our list of spare parts is now closed.

The  $\# B^3(G)$ ,  $\# B_a^3$  mentioned above are factorized as follows

$$\# B^3(\text{ghost}) = \frac{1}{2} B^3(\text{ghost})(-) \cup \frac{1}{2} B^3(\text{ghost})(+),$$

and see here the Figure 49. Here are some facts to be kept in mind.

(146.1) By construction, the COLOUR-CHANGING cylinder  $B^3 \times [r, b]$  does NOT communicate with the outer world of  $Z^4(t = j - 2\varepsilon)$  through the ghostly  $\# B^3$ 's. And then, in the same vein,

(146.2) NO curves in the cylinder stay hooked at the  $\# B^3(\text{ghost})$  (something which would be quite an inconvenience for performing our next STEP II (the actual COULOUR-CHANGE)).

**Explanation:** Look at Figure 39. At the level of Figure 46 we are like in the Figure 39-(I), with a green  $C$  going through  $B_a^3$ , which is like our present moving 1-handles  $G$  (NON-lava) or  $B_a^3$  (LAVA), moving over

$r(j)$ . Once the 1-handle slides, and we move from Figure 39-(I) to the Figures 39-(II and III), the  $A_1$  is now like our  $\#(G \text{ or } B_a^3)$  (ghost), and one can see in these last two Figures that there is now no curve going through the ghostly sites  $A_1$ . Anyway, Figure 39 can serve here as an illustration. The green curve is not hooked at  $A_1$  in (II), (III). A more complex choreography of sliding handles and curves dragged along could be put up, corresponding to the real-life situation of all the  $\# B^3$  (ghost). End of (146.2).

(147) But the colour changing cylinder  $B^3 \times [r, b]$  does communicate with the outside world, not only through its ends, but also through the following sites:  $b^3(\beta)$  which is **not** factorized, and which lives completely on the  $\oplus$  side, the  $b^3(Q, Q')$  which are factorized, but only the  $\oplus$  is explicitly drawn in the Figure 49. Then  $b^3(Q, Q')$ 's are attaching zones of short 1-handles like in Figure 32, cut along their  $h \in R \cap B$  which are mute in our very present story. The  $\{\text{little } C_i\}$  which is seable in the Figure 49 should be compared with the red line in Figure 45-(III). On the  $\oplus$  side we find the  $\sum_j c(r_j) + \sum_i \{\text{little } C_i\} \subset \frac{1}{2} B^3(+)\times [r, b]$ , which can get out of the colour-changing cylinder only through the sites of type  $b^3(\beta)$ ,  $b^3(Q \text{ or } Q')$ . End of (147).

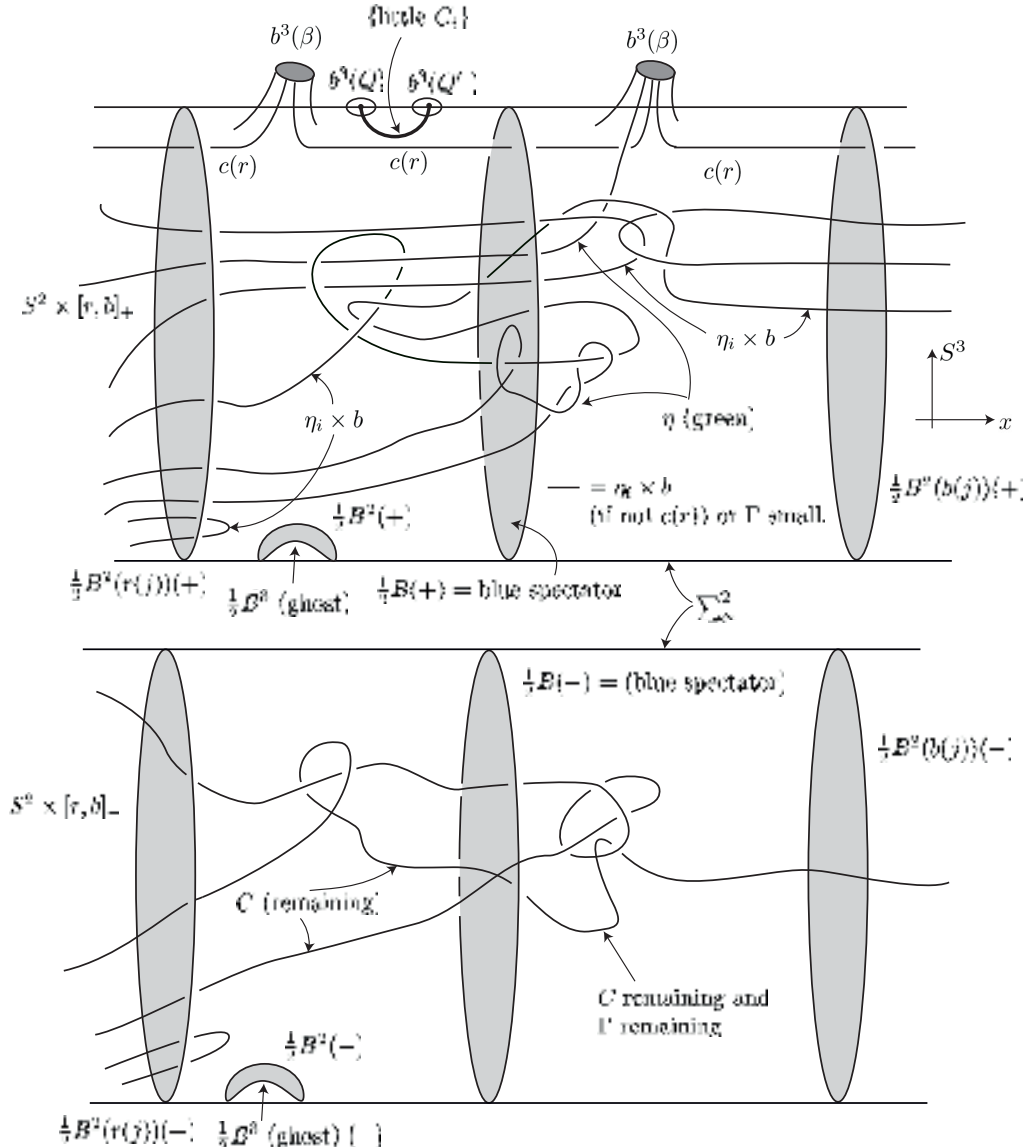


Figure 49.

The COLOUR-CHANGING CYLINDER. In this cylinder the curves in the  $(\pm)$ -halves  $S^2 \times [r, b]_{\pm}$  are **not entangled** with each other, and therefore they can be treated independently. The contacts  $(\Sigma c(r) + \Sigma \eta_i \times b) \cap \frac{1}{2} B^2(r(j)(+))$  are GLUEING TERMS, like in (144.3). [The  $h_j(r) \approx B^3(r(j))$  occurs in Figure 45-(I), where we can also see its GLUEING TERM with  $c(r)$ . The glueing term with  $\eta_i \times b$  has been drawn for pedagogical purposes.] The GLUEING TERMS may be severed and reglued somewhere else on  $\partial \text{LAVA}_r$ , without any harm.

Also, inside  $S^3 \times [r, b](-)$  we have LAVA bridges, which we have not drawn here. They will follow the general fate of the curves. Here  $1/2 B^2(b(j))(-) \cup 1/2 B^2(b(j))(+) = \partial B^3(b(j))$ .

In terms of the coordinate system from Figure 49, our STEP II, which is part of the bigger step  $j - 1 \Rightarrow j$ , is a translation move to the right, expressible as  $x \mapsto x + [r - b]$ , which, in a factorized manner, meaning operating in such a way that the  $\pm$ -SPLITTING should be respected, **superposes** in Figure 49 the  $\frac{1}{2} B^3(r(j))_{\pm}$  on top of  $\frac{1}{2} B^3(b(j))_{\pm}$ , i.e.  $S^3(r(j))$  on top of  $S^3(b(j))$ . At the level of  $Z^4(j - 2\varepsilon) \Rightarrow Z^4(j - \varepsilon)$  (with  $t = j - \varepsilon$  the time when the COLOUR-CHANGING step is completed), the equality of sets  $r(b) = b(j) \subset Z^4(j - \varepsilon)$  has been established. The “site”  $r(j)$  will be forgotten from now on. In more detail, we proceed as follows as far as this “isotopic move of 1-handle cocores” is concerned.

If the word “isotopic” occurs here between quotation marks, it is that during the STEP II, the  $B^3(r(j))$  which moves, inside the COLOUR-CHANGING CYLINDER, into the position  $B^3(b(j))$ , sweeps through  $b^3(\beta)$ ,  $b^3(Q, Q')$  and the  $B^3(\text{ghost})$ . So, when all the rest of the world of  $\Gamma(2\infty)$  is taken into account, our STEP II is certainly NOT **an isotopy**, of anything. It is only a non-isotopic geometric transformation corresponding to the abstract equality  $\Phi(r(j)) = b(j)$  in the Lemma 14. What our STEP II achieves is to force the equality of sets  $B^3(r(j)) = B^3(b(j))$ , keeping the  $B^3(b(j))$  in place as it is. Next, we also have the following items.

•) We do not budge anything else on the  $S^3 \times [r, b](+)$  side, where all the curves stay put, as they are, and the  $\text{LAVA}_b$  too. The only move on the  $\oplus$  side is the  $r(j) \Rightarrow b(j)$  itself.

••) [Important Remark: Of course, our transformation  $r(j) \Rightarrow b(j)$  is, from the viewpoint of the mere COLOUR-CHANGING CYLINDER, Figure 49 is simple-minded isotopy of 1-handle cocores, moving  $B^3(r(j))$  to the right until it is superposed on  $B^3(b(j))$ . But since this **sweeps** through the sites  $b^3(\beta)$ ,  $\# B^3(\text{GHOST})$  it cannot be in any way an isotopy. It is a way to put flush and bones on Lemma 14, actually on the  $\Phi(r(j)) = b(j)$ , and we will **decree** that  $B^3(r(j))$  leaves  $\text{LAVA}$ , while  $B^3(b(j))$  enters it. But the metaphor of translation move to the right, is very convenient for describing the fate of the objects in the  $\pm$  sides of the colour-changing cylinder.] With this, on the  $S^3 \times [r, b](-)$  side, all the {curves} + (LAVA BRIDGES  $\subset \text{LAVA}_r$ ) **move solidarily** with  $\frac{1}{2} B^3(r(j))(-)$  and hence get glued to  $\frac{1}{2} B^3(b(j))(-)$  at the end of the process, and they stick there from there on. This movement is made possible, without any obstructions, because the ghostly  $B^3(-)$ ’s are untouched by the curves; this has been explained in (146.2). Notice, also, that the conjunction of •) and ••) is made possible by the combination

#### SPLITTING + CONFINEMENT.

The topologies of both the individual  $\text{LAVA}_r$  and  $\text{LAVA}_b$  stay intact, but we may sever GLUEING TERMS like  $((\eta_i \times b) + c(r)) \cap \frac{1}{2} B^3(r(j))(+)$ , where  $\frac{1}{2} B^3(r(j))(+) \subset \text{LAVA}_r$ , **decreeing** that  $((\eta_i \times b) + c(r)) \cap \frac{1}{2} B^3(b(j))(+)$  are now NEW GLUEING TERMS. Also, now the  $B^3(b(j)) = \frac{1}{2} B^3(b(j))(-) \cup \frac{1}{2} B^3(b(j)(+))$  becomes  $\text{LAVA}_r(t = j - \varepsilon)$ , by decree too. Here, of course, as already said,  $\{t = j - \varepsilon\} \equiv \{\text{The time when STEP II inside } j - 1 \Rightarrow j \text{ has been completed}\}$ .

In our move on the  $S^3 \times [r, b](+)$  side, we do not worry about severing connections  $\text{LAVA}_b \cap B^3(r(j))$ , these are just freely movable GLUEING TERMS  $\text{LAVA}_b \cap \text{LAVA}_r$ .

It should be stressed that the transformation  $r(j) \Rightarrow b(j)$  operates at the level of the 1-handle cocores themselves, and not at the level of {extended 1-handle cocores}.

Here the  $r(j)$  may be the one in Figure 48 and one should also keep in mind that in the context of the Figure in question, the  $C \cdot h = \text{id} + \text{nilpotent}$  might be bumped. But then, also, the  $r(j)$  drags along all its  $\{\text{LAVA}_r\} + \{\text{the corresponding LAVA bridges}\}$ , and puts them on top of  $b(j)$  which, by degree, becomes itself LAVA.

The  $\text{LAVA}_r$  stays unchanged, up to homeomorphism and hence it keeps its PRODUCT structure intact. The  $\frac{1}{2} B^3(r(j))(+)$  has never contributed to the  $\text{LAVA}_b$  and hence the  $\frac{1}{2} B^3(b(j))(+)$  will not contribute to it either. But these  $B^3$ 's contribute to  $\text{LAVA}_r$  of course, at the appropriate times. [And, at the level of Figure 49, inside the  $S^2 \times [r, b]_+$ , the only contributions to  $\text{LAVA}_r$  can only come from  $\frac{1}{2} B^3(r(j))(+)$  OR from  $\frac{1}{2} B^3(b(j))(+)$ ; they do not occur simultaneously, of course.]

The  $\text{LAVA}_b$  stays intact and then, because of Lemma 18, the global LAVA continues to have its product structure too. Here are other features of our step, as described above.

(147.1) No new contacts  $b(j) \cdot \sum_1^M \eta_\ell(\text{green})$  are introduced. [Remember also that the  $H_{j \leq M}^b$  with their diagonal contacts are all in the passive part  $Y_j$  of Figure 46.]

(147.2) The set of the parasitical  $h_k$ 's in (144.1) can only decrease, i.e. we have no new  $R - B \ni h_i \subset \text{LAVA}(t)$ , touched by  $\Sigma \eta_\ell(\text{green})$ , with respect to what we had already before.

(147.3) With our  $\{\text{extended cocores}\}$  simply defined now by appealing to the PRODUCT PROPERTY OF  $\{\text{LAVA}, \delta \text{LAVA}\}$ , and see here the all-important (144.2) too, any possible inclusion

$$B^3(r(j)) \subset \sum_{j=1}^M \{\text{extended cocore } H_j^b\},$$

has been replaced by a harmless inclusion

$$B^3(b(j)) \subset \sum_{j=1}^M \{\text{extended cocore } H_j^b\}.$$

The point here is that while with  $B^3(r(j)) \subset \{\text{extended cocore } H_j^b\}$  can come with its dangerous contacts  $\eta_\ell(\text{green}) \cdot B^3(r(j)) \subset \eta_\ell(\text{green}) \cdot \{\text{extended cocore } H_j^b\}$ , the LITTLE BLUE diagonalization forbids the contacts

$$\eta_\ell(\text{green}) \cdot B^3(b(j)) \neq \emptyset.$$

(147.4) Since on the  $S^3 \times [r, b](+)$ -side of the COLOUR-CHANGING cylinder all the curves have to stay put, the undesirable (145.6) does not create any new unwanted terms

$$\eta_\ell(\text{green}) \cdot b(j) \subset \eta_\ell(\text{green}) \cdot \{\text{extended cocore } H_j^b\}.$$

So the (145.6) has been taken care of, more precisely the (145.6) becomes *irrelevant* once  $r(j)$  has been replaced by  $b(j)$ .

(147.5) But there is also a piece to be paid for all this. All the  $B \cap X_j^0$  (see (145.4-bis) and Figure 46) are prospective passive BLUE spectators  $b$  for our move  $\{B_a^3(\text{LAVA}) \text{ SLIDES over } B_p^3(\text{LAVA})\}$ , coming with unwanted contacts like (145.5). And these  $b$ 's could happily be among the  $\{\text{past or future } b(j)\text{'s}\} \cap \sum_1^M \{\text{extended cocore } H_j^1\}$ . So the (145.5) *has* to be demolished. This will be achieved by STEP III with which our  $j - 1 \Rightarrow j$  ends. This will also enforce the important condition, part of the LITTLE BLUE DIAGONALIZATION

$$(147.6) \quad \eta_\ell(\text{green}) \cdot \left( B - \sum_{j=1}^M H_j^b \right) = 0.$$

At this point, we will ANTICIPATE a bit. The STEP III inside our big  $j - 1 \Rightarrow j$ , soon to be described explicitly, achieves among other things the following items:

- ) It destroys the (145.5) which the STEP I inside the same  $j - 1 \Rightarrow j$  has created.
- ) It stays far from  $b(j)$  and does not bring any  $\eta_\ell(\text{green})$  on top of the  $b(j)$  in question.
- ) It preserves the PRODUCT PROPERTY of LAVA and also the (144.2). Hand in hand with this, comes the next item.
- ) It does not increase the set of parasitical  $h_k \in R - B$  touched by the  $\eta(\text{green})$  (144.1).

At the beginning of each step  $j - 1 \Rightarrow j$  we assume, inductively, that (147.6) is satisfied, then with the things said it is also verified at the end of  $j - 1 \Rightarrow j$ .

With this, assuming the STEP III inside  $j - 1 \Rightarrow j$  has all the stated virtues, when we get to time  $t = 1$  the (147.6) is with us. Also, the totality of the STEPS II realizes condition (144.1) and hence, also, all the parasitical RED terms in the geometric intersection matrix

$$\eta_i(\text{green}) \cdot \{\text{extended cocore } H_j^b\}, \quad \text{when } 1 \leq i, j \leq M$$

have disappeared. But to get from this to the desired grand BLUE diagonality

$$\eta_i(\text{green}) \cdot \{\text{extended cocore } H_j^b\} = \delta_{ij},$$

we certainly need the (147.6) too. This will make sure that there are no  $b(j - 1) \subset \{\text{extended cocore } H_{i \leq M}^p\}$ , coming with  $\eta_\ell(\text{green})$  on top.

**Step III.** The aim of the present step is two-fold: we want to restore  $N^4(\Gamma(2\infty))$  as it was before Step I i.e. we want to find back our Figure 46, as it stood initially, but of course now with  $r(j - 1)$  a mere “site” without any other geometrical meaning. Our LAVA ( $j - \varepsilon$ ) contains now the  $B^3(b(j))$  in lieu of  $B^3(r(j))$ . And then, we also want to repair the damage (145.5). For that purpose we perform now essentially, the Step I *in reverse*, i.e. schematically

$$\text{The } (t = j - \varepsilon) \xrightarrow[\text{STEP III}]{=} (t = j) \text{ is, essentially, STEP III} = (\text{STEP I})^{-1}.$$

All this is supposed to happen without undoing the colour change  $r(j) \Rightarrow b(j)$ . Since the Step I stayed far from  $b(j)$ , the same is true for STEP III.

Here are more details concerning out STEP III. Although, schematically speaking we perform  $(\text{STEP})^{-1}$ , the  $r(j)$  is now no longer a 1-handle, it is just a “spot”. So, unlike what has happened in the STEP I, we no longer have now slidings of some interesting 1-handles over some other interesting 1-handles, like the kind of things which were displayed in the Figures 40 to 44. As far as the handles are concerned, namely the  $B_a^3(\text{LAVA})$ ’s and the  $G$ ’s, we have now a simple isotopic move without any incidence at the level of the geometric incidence matrices like  $C \cdot h$ . There are no moves of 1-handles sliding over other 1-handles now.

We proceed again in a factorized manner and we want both to destroy the (145.5) and to make sure that the PRODUCT PROPERTY of LAVA continues to be with us.

i) When we are on the  $\oplus$  side, then together with the moving 1-handles  $B_a^3(\text{LAVA})$ , we drag back to their initial (pure Step I) position the external curves  $\eta_\ell(\text{green})$ . This way we dispose of the unwanted (145.5), which disappears, in this process. On the same side  $\oplus$  we also have other contacts  $\{\text{curves}\} \cap (\Sigma B_a^3(\text{LAVA}) + \{\text{Gates } G\})$ , with the  $B_a^3(\text{LAVA}) + \{\text{Gates } G\}$  attached now on the other side of our spot  $r(j)$ , outside of the COLOUR-CHANGING CYLINDER), and these contacts get dragged together with the  $B_a^3(\text{LAVA}) + G$ , back to their initial positions too.

The Step III happens far from  $b(j)$ , on top of which it leaves intact, all the things, curves and LAVA bridges, brought by STEP II.

With all this,  $LAVA_b$  keeps its PRODUCT PROPERTY. We may change around glueing terms, but that is O.K. Of course, in this STEP III, the  $\{G\} + B_a^3(\text{LAVA})$  slide back over the *site*  $r(j)$ , to get back to their original positions.

On the  $\ominus$  side we have both curves ( $\{C(\text{remaining})\}$ 's) and LAVA bridges. STEP II has transformed the contacts

$$(148)-r(j) \quad (\{\text{curves}(\text{remaining})\} + \text{LAVA BRIDGES}) \cap r(j)$$

to isomorphic contacts

$$(148)-b(j) \quad (\{\text{curves}(\text{remaining})\} + \text{LAVA BRIDGES}) \cap b(j).$$

ii) When we are on the  $\ominus$  side and we perform our isotopic move of bringing the  $B_a^3(\text{LAVA}) + G$  back to their original position, we solidarily displace, by dragging along OR pushing in front in a snake-like manner, like in Figure 51-(A)  $\Rightarrow$  51-(B), the curves and LAVA BRIDGES, so that the contacts  $B_a^3(\text{LAVA}) \cap (\text{curves} + \text{LAVA bridges})$  (and similarly for  $G$ , but we do not really care about that) should be isotopically preserved, without bumping (148)- $b(j)$ . This way, the  $LAVA_r$  continues to have its STRONG PRODUCT property.

All this means that the global lava continues to have its product property.

**Final Remark.** In all this story, the BLUE spectators stay passive, curves do not stick to them, even when touching them. [The “sticking” refers here to connections, inside  $LAVA_r$  or  $LAVA_b$ , which when undone destroy the PRODUCT PROPERTY.] Also, and importantly, the curves  $\eta_\ell(\text{green})$  do not touch the blue spectators any longer, at the end of the STEP III. It can be checked that the STEP III, as described above, has the features  $\bullet$ ),  $\bullet\bullet$ ),  $\bullet\bullet\bullet$ ),  $\bullet\bullet\bullet\bullet$ ), listed immediately after the formula (147.6).

**Lemma 20. (The time  $t = 1$  compactification.)** *Proceeding with steps  $(t = j - 1) \Rightarrow (t = j)$  for all the successive times in (144.3) and with each of these  $j - 1 \Rightarrow j$  divided itself into its intermediary STEPS I, II, III on the lines just described, we have inside  $N_1^4(2X_0^2)^\wedge$  a transformation*

$$(148) \quad (N^4(2\Gamma(\infty))) \left( t = \frac{1}{2} \right), N^4(\Gamma(3)) \Longrightarrow (N^4(2\Gamma(\infty))(t = 1), N^4(\Gamma(3)),$$

which is the identity on  $N^4(\Gamma(3))$ , with the following features.

1) *This transformation drags along the link from (101.1-bis), both the internal and the external curves, and also the LAVA bridges too; hence it comes with a transformation which conserves the PRODUCT PROPERTY*

$$(149) \quad LAVA \left( t = \frac{1}{2} \right) \Longrightarrow LAVA(t = 1).$$

2) *The analogue of (137) is valid at  $t = 1$  too, i.e. we have*

$$(150) \quad \left\{ \underbrace{\left[ N^4(2\Gamma(\infty))(t = 1) - \sum_1^\infty h_n(t = 1) \right] \cup LAVA(t = 1)^\wedge}_{\hat{Z}(t=1)} \right\} \sum_1^M \{ \text{extended cocore } H_i^b \}^\wedge =$$

$$\underset{\text{DIFF}}{=} \underset{i=1}{\#}^M (S_i^1 \times B_i^3, (*) \times B_i^3).$$



The analogue of (136) is also valid at  $(t = 1)$ ,

$$(151) \quad \Delta_{\text{Schoenflies}}^4 \stackrel{\text{DIF}}{=} N^4(2X_0^2)^\wedge(t = 1) = \left[ N^4(2\Gamma(\infty))(t = 1) - \sum_1^\infty h_n(t = 1) \cup \text{LAVA}(t = 1)^\wedge \right] + \\ + \sum_1^{\bar{n}} D^2(\Gamma_j) \subset N_1^4(2X_0^2)^\wedge(t = 1) = N^4(2X_0^2)^\wedge(t = 1) \cup \partial(N^4(2X_0^2)^\wedge(t = 1)) \times [0, 1] \stackrel{\text{DIF}}{=} \Delta^4 \cup (\partial\Delta^4 \times [0, 1]).$$

If we disregard all the special subtleties of the COLOUR-CHANGING process, one goes from (136) to (151) by ambient isotopy and, moreover, under the transformation  $(t = \frac{1}{2}) \Rightarrow (t = 1)$ , the  $\sum_1^{\bar{n}} D^2(\Gamma_j)$  does not change otherwise than being dragged via the covering isotopy theorem. These things just said, should be enough for the proof of (151).

3) We have now, for  $1 \leq i, j \leq M$  the BIG BLUE DIAGONALIZATION:

$$(152) \quad \eta_i(\text{green}) \cdot \underbrace{\{\text{extended cocore } H_j^b\}^\wedge}_{\text{as defined by the LAVA } (t = 1)^\wedge} = \delta_{ij}$$

and, of course, we continue to have also

$$\eta_i(\text{green}) \cdot \left( B - \sum_1^M H_j^b \right) = 0.$$

Combining this time  $(t = 1)$ -Lemma with the very first Lemma 3 we have our proof that  $N^4(\Delta^2)$ , albeit now presented as

$$N^4(\Delta^2) \stackrel{\text{DIF}}{=} \left[ \left( N^4(2\Gamma(\infty))(t = 1) - \sum_1^h h_i(t = 1) \right) \cup \text{LAVA } (t = 1)^\wedge \right] + \sum_1^{\bar{n}} D^2(\Gamma_j)$$

is **geometrically simply-connected**. So our Theorem 2, and then Theorem 1 too, are by now proved.

**Additional explanations concerning the change  $(t = \frac{1}{2}) \Rightarrow (t = 1)$ .**

A) In the context of the Figure 46, we certainly have  $\Gamma(3) - R(j) \subset Y_j$ , but this is not necessarily true for the  $\sum_1^{\bar{n}} \Gamma_j$  (see here the PROMOTION TABLE, after the (116)). On the other hand the  $\{\text{little } \Gamma_j\}$ 's may happily occur like the  $\{\text{little } C\}$ 's in Figure 49 and this is without consequence, but, in the context of the  $\{\{\text{Figures 40, 44}\} \text{ for } B_a^3 = G(\text{non LAVA})\}$  the  $\Sigma \{\Gamma_j \text{ remaining}\}$  may well occur too. We treat them like in the usual Figures 40 to 44, they live on the  $\ominus$  side, and they do not stick on any of the 1-handles which are now involved. That means that they can be just ignored as far as LAVA is concerned. Otherwise we drag them along and then finally put them back as they were at  $t = \frac{1}{2}$ , together with the other curves.

B) Clearly, the  $B_a^3(\text{LAVA})$  which has to stick over  $r(j)(\text{LAVA})$  in the Figure 47 is in  $R(j - 1) - \{r(j)\} - B$  and using  $\{G\}$  makes that only a finite subfamily of  $R(j - 1)$  is concerned.

C) When it comes to STEP II, in its translation move to the right along the  $x$ -axis (in Figure 49) the  $r(j)$  which moves to the position  $b(j)$  has, of course, to brush the sites  $b^3(\beta), b^3(Q)$ , and the ghostly  $B^3$ 's. This brushing on the  $\oplus$  side, leaves the curves  $c(r)$ ,  $\{\text{little } C\}$  in place and is without further consequence. Actually, to be precise, the  $b^3(Q)$  have, just like the  $B^3$  (ghost) a  $\oplus$  and a  $\ominus$  half. Only the  $\oplus$  half has been explicitly drawn. As one can see in Figure 45, the  $b^3(\beta)$ 's live completely inside  $N_+^4(2\Gamma(\infty))$  and they are not factorized.

D) We will explain now the factorized little detail

$$\tilde{S}^2 = \frac{1}{2} \tilde{B}^2(-) \cup \frac{1}{2} \tilde{B}^2(+)$$

which occurs in the Figure 47 surrounded by thick dotted lines ( $= - - -$ ) and which bounds the 3-ball:

$$\tilde{B}^3 = \frac{1}{2} \tilde{B}^3(-) \cup \frac{1}{2} \tilde{B}^3(+) \subset \partial N^4(2\Gamma(\infty)),$$

which occurs, also, in the Figure 50 below. This is the place via which the cylinder which connects  $B_p^3$  to  $A_1^3$  at the level of the Figure 47, connects with the COLOUR-CHANGING CYLINDER (145.7), after the  $B_a^3$  (LAVA) has slid over  $B_p^3$  (LAVA). This is displayed, with some detail, albeit schematically, in Figure 50.

After that move,  $B_p^3 = r(j)$  leaves the space of the Figure 47 and, via  $\tilde{S}^2$ , which is actually a section of (146) (lateral surfaces of the COLOUR-CHANGING cylinder), enters the COLOUR-CHANGING cylinder in question and makes its move  $r(j) \xrightarrow{\text{STEP II}} b(j)$ . [In terms of (146) the  $\tilde{S}^2$  is located between  $S^2 \times r$  and  $S^2 \times b$ .]

Coming back to STEP II, our COLOUR-CHANGE is NOT a 1-handle slide, coming globally with some global isotopic move of an  $N^4(2\Gamma(\infty))$ -like object (some  $Z^4(t)$ ), but an internal isotopic repositionning of a 1-handle cocore, in a fixed background.

E) The condition  $C \cdot h = \text{id} + \text{nilpotent}$  which we had at time  $t = \frac{1}{2}$  has gotten bumped as we saw, and the {extended cocores  $H_j^b$ }’s are defined now by appealing directly to the STRONG PRODUCT PROPERTY of LAVA. And, inside the big step  $t = \frac{1}{2} \implies t = 1$ , we have a lot of intermediary smaller steps  $j - 1 \Rightarrow j$  each of them divided into a

STEP I + STEP II + STEP III.

The STEP I and STEP III do not modify the {extended cocore  $H_j^b$ } which they find (at the beginning time  $t = j - 1$  or just after STEP II ( $t = j - \varepsilon$ )). But STEP II achieves the following at  $t = j$ : The set  $\sum_{\ell=1}^M \{\text{extended cocore } H_\ell^b\} \cap \{\text{the parasitical 1-handles from (144.1), at time } t = j\}$ , NO LONGER CONTAINS the  $r(j)$  and this  $r(j)$  will never reappear again inside any of the {extended cocore  $H_\ell^b$ },  $\ell \leq M$ . So, when we get to  $t = 1$  and **all** the STEPS II will have been put into effect (and see here the (144.2) too), then we get that  $\sum_{\ell=1}^M \{\text{extended cocore } H_\ell^b\} \cap \{\text{all the parasitical } h_k \in R - B, \text{ and see here (144.1)}\} = \emptyset$ , actually the  $h_k$ ’s in question have all turned BLUE. This, together with the (147.6) leads to the BIG BLUE DIAGONALIZATION.

F) Here are some additional comments concerning the COLOUR-CHANGING STEP II, from Figure 49, inside  $j - 1 \Rightarrow j$ . We certainly have

$$\{B^3 \times [r, b]\} \cap R(j - 1) \equiv \{\text{our } r(j)\}.$$

At the beginning of STEP II, this  $r(j)$  is LAVA ( $t = j - 1$ ) and, at the end, the  $t = j - \varepsilon$ , it remains a mere ghostly spot which is non LAVA, while  $b(j) \subset \text{LAVA}(t = j - \varepsilon)$ . This  $b(j)$  becoming LAVA at  $t = j - \varepsilon$ , is actually a **decree**. Remember that  $t = j - \varepsilon$  is the time when the STEP II has been completed, and the STEP III, the last piece of  $j - 1 \Rightarrow j$  is ready to start. The STEP III does not touch any longer the composition of the {extended cocore  $H_j^b$ }, which has changed during the STEP II, and so there is no worry concerning the E) above.

G) In all our story, inside the  $Z(t)$ ’s ( $\approx “N^4(2\Gamma(\infty))”$ ) we have strands  $\{C \text{ remaining}\}$ , with  $\{D^2(C)\}$ ’s  $\subset \text{LAVA}$  sticking out of them, possibly glued on LAVA BRIDGES (+ DILATATIONS) and also strands  $\{\Gamma_j \text{ remaining}\}$ , with  $D^2(\Gamma_j)$ ’s, which are NON-LAVA sticking out. The  $\Gamma_i$ ’s stick on the 1-handles of  $\Delta^4$ ,

but they are never glued to the LAVA bridges or dilatations. They run closely and parallel to their mates  $C$  (and LAVA bridges) in the same sheaf.

H) When one looks at Figure 46, at time  $t = j - 1$  both  $\Gamma - R(j - 1)$  and  $\Gamma - B$  are trees. At  $t = j$  the  $R(j) = R(j - 1) - \{r(j)\} + \{b(j)\}$  appears, and now  $\Gamma - R(j)$  and  $\Gamma - B$  are trees. This is the global view. Now, our COLOUR-CHANGING Step II is NOT a 1-handle sliding, like for instance the one in the Figures 47 + 48 but an “isotopy” of 1-handle cocores, an internal and not an external process. In order to be able to realize this, we have to go to  $Z^4(t = 1 - 2\varepsilon)$ , and to sever the connections with the outer world at the site  $b^3(\beta)$ , which are now to be brushed through, in the COLOUR-CHANGING cylinder which we have rather artificially created. Our “isotopy” of 1-handle cocores  $r(j) \Rightarrow b(j)$ , which clearly cannot take place, as such, in the real world, is a convenient recipe for changing in an appropriate way the connections between 1-handles and (2-handles) + LAVA BRIDGES, so as to give consistency to the *colour-changing decree*.

I) Figure 50, which is a companion to Figure 49 is a schematical representation of the creation of our COLOUR-CHANGING cylinder.

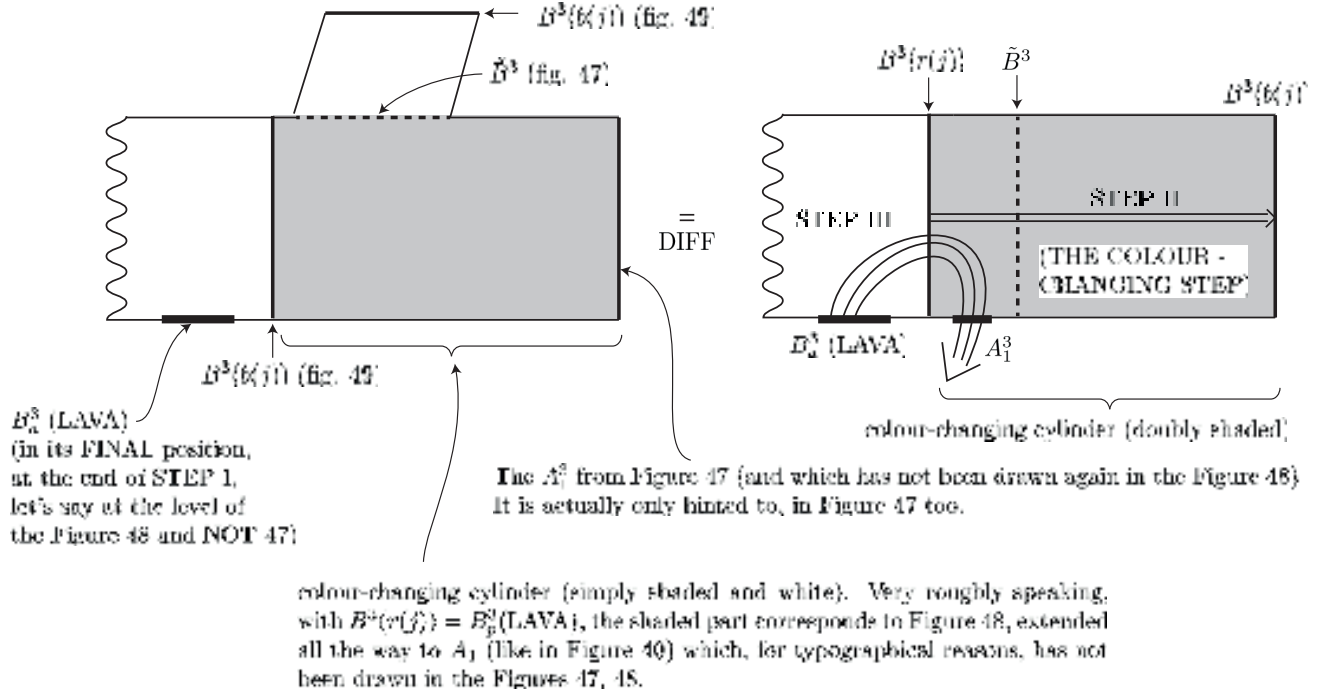


Figure 50.

Geometry of the colour-changing cylinder (A schematical explanation). We can also explain now the  $\tilde{B}^3$  which occurs as a fat dotted lines (- - -) here, and also in Figure 47. Consider Figure 46 and let us say that the  $B_a^3(\text{LAVA})$  in Figure 47 is  $\{(B_a^3)' \text{ OR } (B_a^3)''\}$ , which has gotten now on the same edge as the 1-handle cocore  $r(j)$  and is ready to slide over it (a 1-handle sliding move). Then, on the road from  $(B_a^3)'$  to  $r(j)$  we find the  $U$  (Figure 46), on the way, separating let us say,  $r(j) + (B_a^3)'$  from  $b(j)$ . Then, for  $B_a^3$  (Figure 47) =  $(B_a^3)'$ , the  $U$  is the  $\tilde{B}^3$ . When it comes to  $(B_a^3)''$  or  $(B_a^3)'''$  in Figure 46, it is  $V$  which separates now  $r(j) + (B_a^3)'''$  from  $b(j)$ , and this is now the  $\tilde{B}^3$  in the corresponding Figure 47.

J) We will talk now about STEP III which restores things geometrically (the moving  $Z^4(t)$  is changed back into  $N^4(2\Gamma(\infty))$ ), staying all the time far from  $b(j)$ . Figure 51 suggests what step III is supposed to do to the  $B_a^3(\text{LAVA})$  as it stands after Step I, in its final position from the Figure 48, and it certainly does not budge from there, during the STEP II. The move of  $B_a^3(\text{LAVA})$  sliding over  $(B_p^3 = r(j))(\text{LAVA})$ , which we consider here, is the one from the Figures 47 + 48.

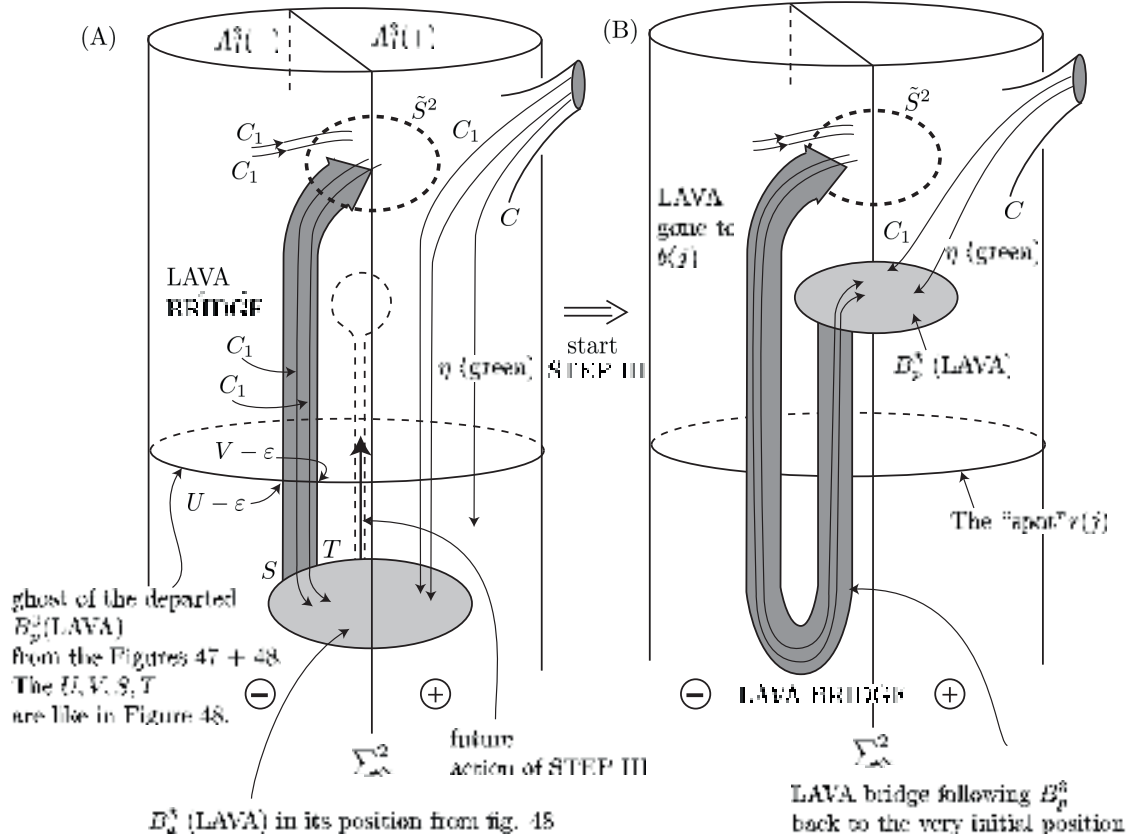


Figure 51.

The STEP III of  $j - 1 \Rightarrow j$  at the level of Figure 48. At the level (A). The STEP II (following the Figure 47  $\xrightarrow{\text{STEP II}}$  48) has been performed, but not yet the STEP III. The  $B^3(p) = r(j)$  from Figure 48 has departed towards the position  $b(j)$ , going through  $\tilde{S}^2$ , with the LAVA BRIDGE dragged along after it. The  $U, V$  from Figure 48 have gone with  $B_p^3$  through  $\tilde{S}^2$ , leaving only the ghostly  $U - \epsilon, V - \epsilon$  behind them. At the level of (B) the STEP III for  $B_a^3$  has just started, taking it, for the time being just over the spot  $r(j)$ . We have tried to render graphically, the contorsion of the LAVA bridges. In (A) we see, dotted, the future move from (B).

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